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IMPLICATIVE STRONG FILTERS IN HYPER QUASI-ORDERED RESIDUATED SYSTEMS

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ABSTRACT. Quasi-ordered residuated systems as a generalization of both quasi-ordered commutative residuated lattices and hoop-algebras was developed in 2018 by Bonzio and Chajda. The ideas of the theory of hyper structures were applied to this algebraic structure by this author and, at the same time, developed the concepts of filters in a hyper quasi-ordered residuated system. In this paper, the conditions that determine the concept of implicative strong filters in it. Some equivalent conditions were found that also determine this concept.

1. INTRODUCTION

Quasi-ordered residuated system is a commutative residuated integral monoids ordered under a quasi-order, introduced by S. Bonzio and I. Chajda in [1]. In the last few years, the theory of quasi-ordered residuated systems was enriched with more results on ideals and filters in them (for example, see [9, 12, 13, 16, 17]). This algebraic structure is a generalization of both commutative residuated lattices and hoop-algebras.

Hyper structure theory was introduced in 1934 in [8], when F. Marty at the 8th congress of scandinavian mathematicians, gave the definition of hypergroup and illustrated some applications and showed its utility in the study of groups, algebraic functions, and rational fraction. Till now, the hyper structures have been studied from the theoretical point of view for their applications to many subject of pure and applied mathematics. Some fields of applications of the mentioned structures are lattices, graphs, coding, ordered sets, median algebra, automata, and cryptography (see, for example [7]). Many researchers have worked on this area. For example, R. A. Borzooei et al. in [2, 3] introduced and studied hyper residuated lattices.

The application of the theory of hyper structures to quasi-ordered residuated systems is the subject of the article [18]. There, among other things, the concept of (strong) deductive systems and the concept of (strong) filters in a hyper quasi-ordered residuated system were developed.

After the introduction, in the Preliminaries section, necessary material is presented for a comfortable follow-up of the material in Section 3, which is the main part of this paper. I Preliminaries section has three subsections. In the first subsection, the potential reader

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is offered a few words about hyper structures, while in the second subsection, the necessary material on quasi-ordered residuated systems is presented. In the last subsection, the necessary definitions and some assertions about hyper quasi-ordered residuated systems are presented, mostly taken from the article [18]. In Section 3, the author deals with designing the conditions that determine the concept of implicative strong filters in this hyper structure (Definition 3.1) . Then, it was shown that an implicative strong filter in a hyper quasi-ordered residuated system is a strong filter in it (Theorem 3.6). Some equivalent conditions were found that also determine this concept (Theorem 3.8). Of course, it was shown that a strong filter does not have to be an implicative strong filter (Example 3.4).

Issues not treated in this report are the determination of the concept of implicative filters in a hyper quasi-ordered residuated system and its relation to the concept of implicative strong filters in this class of algebraic structures.

2. PRELIMINARIES

In this section, the necessary notions and notations and some of their interrelationships, mostly taken from paper [1, 9, 12, 16], are listed in the order to enable a reader to comfortably follow the presentation in this report. It should be pointed out here that the notations for logical conjunction, logical implication and others have a literal meaning. Thus, for example, the label $H \vDash Q$ has the meaning that the consequent Q can be demonstrated from the hypothesis H. The notation =: in the formula $A =: B$ serves to indicate that A in it is the abbreviation for the formula B.

2.1. **A few words about hyper structures.**

Now, we recall some basic notions of the hypergroup theory from [7]: Let H be a non-empty set. A hypergroupoid is a pair $(H, \circ)$, where $\circ : H \times H \longrightarrow \mathcal{P}(H)\backslash\{\emptyset\}$ is a binary hyperoperation on H. If $a\circ(b\circ c) = (a\circ b)\circ c$ holds, for all $a, b, c \in H$ then $(H, \circ)$ is called a semihypergroup, and it is said to be commutative if $\circ$ is commutative. An element $1 \in H$ is called a unit, if $a \in (1\circ a)\cap(a\circ 1)$, for all $a \in H$ and it is called a scaler unit, if $\{a\} = 1 \circ a = a \circ 1$, for all $a \in H$. Note that if $A, B \subseteq H$, then $A \circ B = \bigcup_{a\in A,\, b\in B}(a \circ b)$.

In addition to the previous one, in what follows the following notations are also will be used ([3]):

Let $(H, \preccurlyeq)$ be a quasi-ordered set and A, B be two subsets of H. Then we write

- $A \ll B$, if there exist $a \in A$ and $b \in B$ such that $a \preccurlyeq b$.
- $A \preccurlyeq B$ if for any $a \in A$, there exists $b \in B$ such that $a \preccurlyeq b$.
- We will write $A \preccurlyeq b$ instead of $A \preccurlyeq \{b\}$.

In light of the foregoing determination, we have $x \preccurlyeq y$ if and only if $\{x\} \preccurlyeq \{y\}$. Also, we will write $a \ll B$ instead of $\{a\} \ll B$.

One can easily conclude that the relation $\preccurlyeq$ is a quasi-order on $\mathcal{P}(H)$. Indeed. Since reflexivity is obvious, let's show transitivity. Let $A, B, C \subseteq H$ be such that $A \preccurlyeq B$ and $B \preccurlyeq C$. Then for any $a \in A$ there exists an element $b = b(a) \in B$ such that $a \preccurlyeq b(a)$ and for any $b \in B$ there exists an element $c = c(b) \in C$ such that $b \preccurlyeq c(b)$. So, for any $a \in A$ there exists an element $c = c(b(a)) \in C$ such that $a \preccurlyeq c$. This means that $A \preccurlyeq C$. In the general case, this relation is not antisymmetric.

Also, it is easy to see that $A \preccurlyeq B \implies A \ll B$. In addition to the previous one, the following applies

$$(\forall a \in H)(\forall b \subseteq H)(a \ll B \iff a \preccurlyeq B).$$

In the special case, for $B = \{b\}$, we have $(\forall a, b \in H)(a \ll b \iff a \preccurlyeq b)$. Finally, let's point out that the following holds

$$(\forall a \in H)(\forall B \subseteq H)(a \in B \implies (a \preccurlyeq B \wedge a \ll B)).$$

Also $\emptyset \neq A \subseteq B \implies B \ll A$ holds for $A, B \subseteq H$. Indeed. $A \subseteq B$ means that $(\forall a \in H)(a \in A \implies a \in B)$ holds. Therefore, one can find $b \in B$ such that $b \in A$. Since $b \preccurlyeq b$, we have $B \ll A$.

2.2. **Quasi-ordered residuated systems.** In article [1], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

Definition 2.1 ([1])**.** A *quasi-ordered residuated system* is a structure $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and $\preccurlyeq$ is a quasi-order relation on A and satisfying the following properties:
(1) $(A, \cdot, 1)$ is a commutative monoid;
(2) $(\forall x \in A)(x \preccurlyeq 1)$;
(3) $(\forall x, y, z \in A)(x \cdot y \preccurlyeq, z \iff x \preccurlyeq y \rightarrow z)$.
We will refer to the operation $\cdot$ as (commutative) multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation.

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 2.1 ([1], Proposition 3.1)**.** *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then*
(4) *The operation '$\cdot$' preserves the pre-order in both positions;*

$$(\forall x, y, z \in A)(x \preccurlyeq y \implies (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y));$$

(5) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (y \rightarrow z \preccurlyeq x \rightarrow z \wedge z \rightarrow x \preccurlyeq z \rightarrow y))$*;*
(6) $(\forall y, z \in A)(x \cdot (y \rightarrow z) \preccurlyeq y \rightarrow x \cdot z)$*;*
(7) $(\forall x, y, z \in A)(x \cdot y \rightarrow z \preccurlyeq x \rightarrow (y \rightarrow z))$*;*
(8) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preccurlyeq x \cdot y \rightarrow z)$*;*
(9) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preccurlyeq y \rightarrow (x \rightarrow z))$*;*
(10) $(\forall x, yz \in A)((x \rightarrow y) \cdot (y \rightarrow z) \preccurlyeq x \rightarrow z)$*;*
(11) $(\forall x, y \in A)((x \cdot y \preccurlyeq x) \wedge (x \cdot y \preccurlyeq y))$*;*
(12) $(\forall x, y, z \in A)(x \rightarrow y \preccurlyeq (y \rightarrow z) \rightarrow (x \rightarrow z))$*;*
(13) $(\forall x, y, z \in A)(y \rightarrow z \preccurlyeq (x \rightarrow y) \rightarrow (x \rightarrow z))$.

It is generally known that a quasi-order relation $\preccurlyeq$ on a set A generates a equivalence relation $\equiv_{\preccurlyeq} := \preccurlyeq \cap \preccurlyeq^{-1}$ on A. Due to properties (4) and (5), this equality relation is compatible with the operations in A. Thus, $\equiv_{\preccurlyeq}$ is a congruence on A. In the light of the previous note, it is easy to see that the following applies: (7) and (8) give:
(14) $(\forall x, y, z \in A)(x \cdot y \rightarrow z \equiv_{\preccurlyeq} x \rightarrow (y \rightarrow z))$.
Due to the universality of formula (9) (or, due to the commutativity of the multiplication, from (14)) we have:
(15) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \equiv_{\preccurlyeq} y \rightarrow (x \rightarrow z))$.
Also, from (11) and (2), it follows
(16) $(\forall x \in A)(x \rightarrow x \equiv_{\preccurlyeq} 1)$
In the general case,
(17) $(\forall x, y \in A)(x \preccurlyeq y \iff x \rightarrow y \equiv_{\preccurlyeq} 1)$
is valid, which is obtained by referring to (11) and (2).

From the previous analysis it can be concluded that a quasi-ordered residuated system is a generalization of a hoop-algebra (in the sense of [5]) because the following formula

$$(\forall x, y \in A)(x \cdot (x \to y) \equiv_{\preccurlyeq} y \cdot (y \to x))$$

does not have to be a valid formula in the observed algebraic structure in the general case. Since the axioms used to determine the hoop-algebra are mutually independent, there must be a model in which the mentioned formula is not valid. So, there is (at least one) example of a quasi-ordered residuated system that is not a hoop-algebra. Several examples of this algebraic structure can be found in previously published papers such as, for example, [9, 10, 12, 14].

A quasi-ordered residuated system $\mathfrak{A}$ is said to be a strong quasi-ordered residuated system ([14], Definition 6) if additionally the following
(18) $(\forall x, y \in A)((x \to y) \to y \equiv_{\preccurlyeq} (y \to x) \to x)$
is valid. If we recall that a hoop is a Weisberg hoop if condition (18) is added to the axioms that determine the concept of hoops, then we can conclude that a strong quasi-ordered residuated system is a generalization of Weisberg hoops. It is well known that the underlying ordering of a Wajsberg hoop is a lattice ordering, and that the join is term-definable by $a \sqcup b =: (a \to b) \to b$. Since any hoop satisfies the equation ([6]) $(a \to b) \to (b \to a) = b \to a$, any Weisberg hoop satisfies the pre-linearity condition $(a \to b) \sqcup (b \to a) = 1$. However, a strong quasi-ordered residuated the system, in the general case, does not have to satisfy the pre-linearity condition. In addition, a strong QRS does not have to satisfy the above-mentioned condition that distinguishes a QRS from a hoop-algebra. Thus, a strong quasi-ordered residuated system is a generalization of Weisberg hoops. Several examples of strong quasi-ordered residuated systems can be found in the articles [14, 15], for example.

The concept of filters in a quasi-ordered residuated system was introduced in the article [9]. This concept is somewhat different from the filter concept in both hoop-algebras and residuated lattices.

Definition 2.2 ([9], Definition 3.1)**.** For a subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a *filter* of $\mathfrak{A}$ if it satisfies conditions
(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$, and
(F3) $(\forall u, v \in A)((u \in F \wedge u \to v \in F) \implies v \in F)$.

Let it note that the empty subset of A satisfies the conditions (F2) and (F3). Therefore, $\emptyset$ is a filter in $\mathfrak{A}$. It is shown ([9], Proposition 3.4 and Proposition 3.2), that if a non-empty subset F of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F2), then it also satisfies the following conditions
(F0) $1 \in F$ and
(F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$.
Also, it can be seen without difficulty that $((F3) \wedge F \neq \emptyset) \implies (F2)$ is valid. Indeed, if (F3) holds, then the formula $u \in F \wedge u \preccurlyeq v$, can be transformed into the formula $u \in F \wedge u \to v \equiv_{\preccurlyeq} 1 \in F$ by (F0) so from here, according to (F3), it can be demonstrate the validity of implications (F2). However, the reverse does not have to be valid.

A reader can find several examples of filters in this algebraic system in the articles [9, 10, 12, 13, 15, 16, 17].

The concept of one recognizable special filter in a quasi-ordered residuated system A is given by the following definition:

Definition 2.3. ([11], Definition 3.1) For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that the implicative filter in $\mathfrak{A}$ if (F2) and the following condition

(IF) $(\forall u, v, z \in A)((u \to (v \to z) \in F \wedge u \to v \in F) \implies u \to z \in)$

are valid.

In the mentioned paper it was shown ([11], Theorem 3.1) that an implicative filter in a QRS $\mathfrak{A}$ is a filter in $\mathfrak{A}$.

2.3. **Hyper quasi-ordered residuated systems.** A hyper quasi-ordered redissued system is introduced by the following definition:

Definition 2.4. ([18], Definition 3.5) A hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A} =: (A, \circ, 1, \to, \preccurlyeq)$, (hyper QRS, by briefly), is a non-empty quasi-ordered set $(A, \preccurlyeq)$ endowed with two binary hyper operations $\circ$ and $\to$ and the element 1 such that satisfying the following conditions:

(H1) $(A, \circ, 1)$ is a commutative semihypergroup with 1 as the unit.

(H0) $(\forall x \in A)(x \in 1 \circ x)$.

(H2) $(\forall x \in A)(x \preccurlyeq 1)$.

(H3) $(\forall x, y, z \in A)(x \circ y \ll z \iff x \ll y \to z)$.

We will denote this system of axioms by $\mathfrak{H}$. With $[\mathfrak{H}]$ we will denote everything that has been demonstrated using these axioms (the so-called the theory developed over these axioms) up to the place of using this notation.

Example 2.5. Any hyper (commutative) residuated lattice (determined as in the article [3], for example) is a hyper quasi-ordered residuated system.

Example 2.6. Any hyper hoop-algebra (determined as in the article [4]) is a hyper quasi-ordered system.

Example 2.7. Let $A = \langle -\infty, 1] \,(\subseteq \mathbb{R})$. Then $(A, \leqslant)$ with the natural ordering is a partially ordered set. Define the hyperoperations $\circ$ and $\to$ on A as follows: $a \circ b =: min\{a, b\}$ and $a \to b =: \{1\}$ if $a \leqslant b$ and $a \to b =: [b, 1]$ if $b < a$. It is not difficult to check that $(A, \circ, 1, \to, \leqslant)$ is a hyper (quasi-)ordered residuated system.

Example 2.8. Let $A = \langle \infty, 1] \,(\subseteq \mathbb{R})$. Define the hyper operations $\circ$ and $\to$ on A as follows:

$(\forall x, y \in A)(x \circ y = \{1, x, y\})$ and

$(\forall x, y \in A)((x \leqslant y \implies x \to y = \{1, y\}) \wedge (y < x \implies x \to y = \{y\}))$.

Then $(A, \circ, 1, \to, \leqslant)$ is a (quasi-)orderd residuated system.

Example 2.9. Let $B =: \{x_i \,:\, i \in \mathbb{N}\}$ and $A = B \cup \{1\}$ with

$$(\forall i \in \mathbb{N})(x_i \neq 1),\ (\forall i \in \mathbb{N})((x_i \preccurlyeq 1) \wedge (x_i \preccurlyeq x_i)) \text{ and } 1 \preccurlyeq 1.$$

Define binary hyperoperations $\circ$ and $\to$ on A as follows:

$$(\forall a, b \in A)(a \circ b =: \{x \in A \,:\, a \preccurlyeq x \wedge b \preccurlyeq x\})$$

and

$a \preccurlyeq b \implies a \to b =: \{1\}$,

$(a = 1 \wedge b \in B) \implies a \to b =: B$,

$(a = x_i \wedge b = x_j \neq a) \implies a \to b =: \{x_k : k \in \mathbb{N} \wedge k \leqslant max\{i, j\}\} \cup \{1\}$

for all $a, b \in A$. With a little more effort, it can be verified that $(A, \circ, 1, \preccurlyeq, \to)$ is a hyper (quasi-)ordered residuated system.

Example 2.10. Let $A =: \{a, b, c, 1\}$ be a chain such that $a < b < c < 1$. Let us defile the hyper operations as follows

$\rightarrow$	1	a	b	c
1	{1}	{a,1}	{b}	{c,1}
a	{1}	{1}	{1}	{1}
b	{1}	{b,c,1}	{b,1}	{1}
c	{1}	{b,1}	{b}	{c,1}

and

$\circ$	1	a	b	c
1	{a,b,c,1}	{a}	{a,b}	{a,b,c}
a	{a}	{a}	{a}	{a}
b	{a,b}	{a}	{a,b}	{a,b}
c	{a,b,c}	{a}	{a,b}	{a,b,c}

Routine calculations show that $(A, \circ, 1, \rightarrow, \leqslant)$ is a hyper (quasi-)ordered residuated system.

The following two propositions list some of the important fundamental features of this hyper system.

Proposition 2.2 ([18]). *In any hyper quasi-ordered residuated system* $(A, \circ, 1, \rightarrow, \preccurlyeq)$*, the following holds:*
(a) $(\forall B \subseteq A)(1 \ll B \implies 1 \in B)$.
(b) $(\forall x, y \in A)(x \preccurlyeq y \implies 1 \in x \rightarrow y)$.
(c) $(\forall x \in A)(1 \in x \rightarrow x)$.
(d) $(\forall x \in A)(1 \in x \rightarrow 1)$.
(e) $(\forall B.C, D \subseteq A)(B \ll C \rightarrow D \iff B \circ C \ll D)$.
(f) $(\forall x, y \in A)(x \circ y \ll x \wedge x \circ y \ll y)$.
(g) $(\forall B, C \subseteq A)(B \circ C \ll B \wedge B \circ C \ll C)$.
(h) $(\forall x, y \in A)(x \ll y \rightarrow x)$.
(i) $(\forall x, y \in A)(1 \in x \rightarrow (y \rightarrow x))$.
(j) $(\forall B, C \subseteq A)(B \ll C \rightarrow B)$.
(k) $(\forall x, y \in A)(x \circ (x \rightarrow y) \ll x \ \wedge \ x \circ (x \rightarrow y) \ll y)$.
(l) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preccurlyeq (x \circ y) \rightarrow z \preccurlyeq x \rightarrow (y \rightarrow z) \preccurlyeq y \rightarrow (x \rightarrow z))$.
(m) $(\forall x, y \in A)(x \ll y \rightarrow (x \circ y))$.

Proposition 2.3 ([18]). *In any hyper quasi-ordered residuated system* $(A, \circ, 1, \rightarrow, \preccurlyeq)$*, the following holds:*
(n) $(\forall x, y, z \in A)(x \preccurlyeq y \implies x \circ z \ll y \circ z)$.
(p) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (z \rightarrow x \preccurlyeq z \rightarrow y \ \wedge \ y \rightarrow z \preccurlyeq x \rightarrow z))$.
(q) $(\forall x, y, z \in A)(x \rightarrow y \preccurlyeq (y \rightarrow z) \rightarrow (x \rightarrow z))$.
(r) $(\forall x, y, z \in A)((x \rightarrow y) \circ (y \rightarrow z) \ll x \rightarrow z)$.
(s) $(\forall x, y, z \in A)(y \rightarrow z \ll (x \rightarrow y) \rightarrow (x \rightarrow z))$.

In this subsection, we will consider the following conditions imposed on the subset F of the hyper quasi-ordered residuated system $(A.\circ, 1, \rightarrow, \preccurlyeq)$:
(HF0) $1 \in F$.
(HF1) $(\forall x, y \in A)(x \circ y \subseteq F \implies (x \in F \wedge y \in F))$.
(HF2) $(\forall x, y \in A)((x \preccurlyeq y \wedge x \in F) \implies y \in F)$.
(HF3) $(\forall x, y \in A)((x \in F \wedge x \rightarrow y \subseteq F) \implies y \in F)$.
(sHF3) $(\forall x, y \in A)((x \in F \wedge F \ll x \rightarrow y) \implies y \in F)$.
(dHF3) $(\forall x, y \in A)(((x \rightarrow y) \cap F \neq \emptyset \wedge x \in F) \implies y \in F)$.
(SH) $(\forall x, y \in A)((x \in F \wedge y \in F) \implies x \circ y \subseteq F)$.
(wSH) $(\forall x, y \in A)((x \in F \wedge y \in F) \implies F \ll x \circ y)$.

Based on the hyper residuated lattices theory (for example, [2]), we introduce the term deductive system in a hyper QRS.

Definition 2.11. Subset D of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ is a deductive system in $\mathfrak{h}\mathfrak{A}$ if it satisfies the conditions (HF0) and (HF3).

In contrast to the mentioned text, here we introduce the term 'strong' deductive system.

Definition 2.12. Subset D of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ is a strong deductive system in $\mathfrak{h}\mathfrak{A}$ if it satisfies the conditions (HF0) and (sHF3).

Definition 2.13. Subset D of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ is a reflexive subset of $\mathfrak{h}\mathfrak{A}$ if it satisfies the following condition

(R) $(\forall B, C \subseteq A)((B \rightarrow C) \cap D \neq \emptyset \implies (B \rightarrow C) \subseteq D)$.

Example 2.14. Let $A =: \{a, b, c, 1\}$ be a chain such that $a < b < c < 1$. Let us defile the hyper operations as follows

$\rightarrow$	1	a	b	c
1	{1}	{a}	{b}	{c}
a	{1}	{1}	{1}	{1}
b	{1}	{a,b}	{1}	{1}
c	{1}	{a}	{a,b}	{1}

and

$\circ$	1	a	b	c
1	{1}	{a}	{b}	{c}
a	{a}	{a}	{a}	{a}
b	{b}	{a}	{a,b}	{b}
c	{c}	{a}	{b}	{c}

Routine calculations show that $(A, \circ, 1, \rightarrow, \leqslant)$ is a hyper (quasi-)ordered residuated system. The subsets $G =: \{1\}$ and $F =: \{c, 1\}$ are reflexive deductive systems in $\mathfrak{h}\mathfrak{A}$.

Example 2.15. Let $\mathfrak{h}\mathfrak{A}$ be a hyper QRS as in Example 2.7. It is easy to show that $D = [\frac{1}{2}, 1]$ is a deductive system in $\mathfrak{h}\mathfrak{A}$.

Example 2.16. Let $\mathfrak{h}\mathfrak{A}$ be a hyper QRS as in Example 2.9. It can be verified that the set $D_n = \{1, x_1, ..., x_n\}$ is a deductive system for any $n \in \mathbb{N}$. However, the set D_n is not a strong deductive system. Indeed, for example, for $x_n \in D_n$ and $D_n \ll x_n \rightarrow x_{n+1} = D_{n+1}$, we have $x_{n+1} \notin D_n$.

Proposition 2.4 ([18], Proposition 3.21)**.** *Let D be a reflexive deductive subset of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$. Then*

$$(\forall B, C \subseteq A)(D \ll B \rightarrow C \iff (B \rightarrow C) \cap D \neq \emptyset \iff (B \rightarrow C) \subseteq D).$$

The concept of filters in a hyper residuated lattice was introduced in [19] and discussed in more detail in [3] as follows: A nonempty subset F of a hyper resuduated lattice L satisfying (HF2) and (SH) is a filter of L. A nonempty subset F of a hyper resuduated lattice L satisfying (HF2) and (wSH) is a weak filter of L. In [3] it was shown (Theorem 3.4) that: A non-empty subset F of a hyper residuated lattice L is a weak filter if and only if it satisfies the condition (HF2) and

(wHF4) $(\forall x, y \in A)((x \in F \wedge y \in F) \implies (x \circ y) \cap F \neq \emptyset)$.

The concept of filters in a hyper hoop-algebra can be found in [4] and is determined in the following way: A nonempty subset F of a hyper hoop-algebra H satisfying (HF2) and (wHF4) is a weak filter of H. A nonempty subset F of a hyper hoop-algebra H satisfying (HF2) and (SH) is a filter of H. In addition, it was shown there: A non-empty subset F of a hyper hoop-algebra H is a weak filter of H if and only if it satisfies conditions (HF2) and (wSH).

In accordance with our earlier orientations - the omission the requires that a filter in a semigroup A be a subsemigroup of the semigroup A (see, for example [9]), we will determine the concept of filters in a hyper QRS as follows:

Definition 2.17. ([18]) A subset F of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ is a filter of $\mathfrak{h}\mathfrak{A}$ if it satisfies the conditions (HF2) and (HF3).

It is not difficult to conclude that $\emptyset$ and A are filters in a quasi-ordered reduced system $\mathfrak{h}\mathfrak{A}$, since the empty set $\emptyset$ and set A satisfy the conditions (HF2) and (HF3).

Definition 2.18. ([18], Definition 3.27) A subset F of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ is a strong filter of $\mathfrak{h}\mathfrak{A}$ if it satisfies the conditions (HF2) and (sHF3).

In [18] it is shown (Theorem 3.24)

$$[\mathfrak{H}], F \neq \emptyset, (HF2) \vDash (dHF3) \iff (sHF3).$$

Example 2.19. Let $\mathfrak{h}\mathfrak{A}$ be as in Example 2.7. For each $a \leqslant 1$, the set $[a, 1]$ is a filter in $\mathfrak{h}\mathfrak{A}$ but it is not a strong filter in $\mathfrak{h}\mathfrak{A}$. Take $x, y \in A$ such that $x \in [a, 1]$ and $y < x$. Then $x \to y = [y, 1]$. If we assume that $x \to y = [y, 1] \subseteq [a, 1]$ holds, then it must be $y \in [a, 1]$. However, for elements $x, y \in A$ from $x \in [a, 1]$ and $(x \to y) \cap [a, 1] = [y, 1] \cap [a, 1] \neq \emptyset$ not must follow $y \in [a, 1]$.

3. Implicative strong filters

This section is the main part of this paper. Here we introduce and analyze the conditions obtained by generalizing in a hyper quasi-ordered residuated system the condition (IF) for mentioned filters in a quasi-ordered residuated system. In order to determine the concept of implicative strong filters in a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$, we will analyze the following conditions:

(dHIF3) $((x \to (y \to z)) \cap F \neq \emptyset \wedge (x \to y) \cap F \neq \emptyset) \implies (x \to z) \cap F \neq \emptyset$
(HIF3) $(x \to (y \to z) \subseteq F \wedge x \to y \subseteq F) \implies x \to z \subseteq F$
(sHIF3) $(F \ll x \to (y \to z) \wedge F \ll x \to z) \implies F \ll x \to z$.

If F is a reflexive subset in a hyper quasi-ordered reduced system $\mathfrak{h}\mathfrak{A}$, then the conditions (dHIF), (HIF3) and (sHIF) are equivalent according to Proposition 2.4. Therefore, in the analysis that follows, we will assume that, in the general case, the subset F is not reflexive.

The first conclusion is given by the following lemma:

Lemma 3.1. $[\mathfrak{H}]$, $F \neq \emptyset$, $(HF2) \vDash (dHIF3) \implies (sHIF3)$.

Proof. Let $\mathfrak{h}\mathfrak{A}$ be a hyper quasi-ordered residuated system and let F be a non-empty subset in A satisfying the conditions (HF2) and (dHIF3). Let us take elements $x, y, z \in A$ such that $F \ll x \to (y \to z)$ and $F \ll x \to y$. This means that there exist elements $s, t \in F$, $u \in x \to (y \to z)$ and $v \in x \to y$ such that $s \preccurlyeq u$ and $t \preccurlyeq v$. Thus $u, v \in F$ by (HF2). hence $(x \to (y \to z)) \cap F \neq \emptyset$ and $(x \to y) \cap F \neq \emptyset$. From here we get $(x \to z) \cap F \neq \emptyset$ according to (dHIF3). Therefore, there exists some $w \in F \cap (x \to z) \neq \emptyset$. The latter gives $F \ll x \to z$ since $w \preccurlyeq w$. This proves the validity of the condition (sHIF3). □

On the other hand, the following lemma gives us another conclusion:

Lemma 3.2. $[\mathfrak{H}]$, $F \neq \emptyset$, $(HF2) \vDash (sHIF3) \implies (dHIF3)$.

Proof. Let $\mathfrak{h}\mathfrak{A}$ be a hyper quasi-ordered residuated system and let F be a non-empty subset in A satisfying the condition (sHIF3). Let us take elements $x, y, z \in A$ such that $(x \to (y \to z)) \cap F \neq \emptyset$ and $(x \to y) \cap F \neq \emptyset$. This means that there exist $u \in (x \to (y \to z)) \cap F$ and $v \in (x \to y) \cap F$. From here we conclude that $F \ll (x \to (y \to z)$ and $F \ll x \to y$ are valid since $u \preccurlyeq u$ and $v \preccurlyeq v$ due to the transitivity of the relation $\preccurlyeq$. Thus $F \ll x \to z$ by (sHIF3). Thus, there exist $t \in F$ and $w \in x \to z$ such that $t \preccurlyeq w$. Hence, $w \in F$ by (HF2). Therefore, $w \in (x \to z) \cap F \cap \emptyset$, which proves the validity of the condition (dHIF3). □

The following definitions introduce the corresponding concept of implicative strong filters in a hyper quasi-ordered residuated system.

Definition 3.1. Let $\mathfrak{h}\mathfrak{A} =: (A, \circ, 1, \rightarrow, \preccurlyeq)$ be a hyper quasi-ordered residuated system and let F be a non-empty subset of A that satisfies the condition (HF2). Then F is called implicative strong filter in $\mathfrak{h}\mathfrak{A}$ if (sHIF) holds.

Relying on Lemma 3.1 and Lemma 3.2, we immediately conclude:

Theorem 3.3. *A nonempty subset F in a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ is an implicative strong filter in $\mathfrak{h}\mathfrak{A}$ if and only if it satisfies the conditions (HF2) and (dHIF3).*

In what follows, we need the following lemmas:

Lemma 3.4. *Let $\mathfrak{h}\mathfrak{A}$ be a hyper residuated system $\mathfrak{h}\mathfrak{A}$. Then*

$$(\forall x \in A)(\exists u \in A)(x \equiv_{\preccurlyeq} u \wedge u \in 1 \rightarrow x).$$

Proof. Let $x \in A$ be an arbitrary element. On the one hand, we have $x \in 1 \circ x$ by (H0). Then, we get $1 \circ x \ll x$ by reflexivity of the quasi-order. Thus $x \ll 1 \rightarrow x$ by (H3). Hence there exists $u \in 1 \rightarrow x$ such that $x \preccurlyeq u$. On the other hand, for arbitrary $u \in 1 \rightarrow x$, we also have $u \ll 1 \rightarrow x$ by reflexivity of the quasi-order. So, $u = 1 \circ u \ll x$. Therefore, $x \preccurlyeq u \wedge u \preccurlyeq x$. This means $u \equiv_{\preccurlyeq} x$. □

Lemma 3.5. *Let F be a non-empty subset of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ which satisfies the condition (HF2). Then:*
(i) $(\forall x \in A)(x \in F \iff (1 \rightarrow x) \cap F \neq \emptyset)$*;*
(ii) $(\forall x, y \in A)((x \rightarrow y) \cap F \neq \emptyset \implies (1 \rightarrow (x \rightarrow y)) \cap F \neq \emptyset)$.

Proof. (i) For $x \in F$, there exists an element $u \in 1 \rightarrow x$ such that $x \preccurlyeq u$ according to the previous lemma. Then $u \in F$ by (HF2). Thus $u \in (1 \rightarrow x) \cap F$. This means $(1 \rightarrow x) \cap F \neq \emptyset$.

Assume $(1 \rightarrow x) \cap F \neq \emptyset$. So, there exists an element $u \in A$ such that $u \in F$ and $u \in 1 \rightarrow x$. Then $u \ll 1 \rightarrow x$ and $u = u \circ 1 \ll x$ by (H3). This means $u \preccurlyeq x$. Thus $x \in F$ by (HF2).

(ii) For $u \in (x \rightarrow y) \cap F \neq \emptyset$, i.e. for $u \in x \rightarrow y$ and $u \in F$, we have $u \circ 1 = u \ll x \rightarrow y$. Then $u \ll 1 \rightarrow (x \rightarrow y)$. This means that rhere exists an element $v \in 1 \rightarrow (x \rightarrow y)$ such that $u \preccurlyeq v$. Thus $v \in F$ by (HF2). So, $u \in (1 \rightarrow (x \rightarrow y)) \cap F$. Therefore, $(1 \rightarrow (x \rightarrow y)) \cap F \neq \emptyset$. □

In the following theorem, we prove that an implicative strong filter in a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ is a strong filter in $\mathfrak{h}\mathfrak{A}$.

Theorem 3.6. *Let $\mathfrak{h}\mathfrak{A}$ be a hyper quasi-ordered residuated system. Then*

$$[\mathfrak{H}], F \neq \emptyset, \ (HF2) \vDash (sHIF3) \implies (sHF3).$$

Proof. Let a non-empty set F in a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ satisfy the conditions (HF2) and (sHIF3). Let us prove that the formula (sHF3) can be deduced from the previous two. To this end, it suffices to prove the validity of the formula (dHF3).

Let $x, y \in A$ be such that $x \in F$ and $(x \rightarrow y) \cap F \neq \emptyset$. Then $(1 \rightarrow x) \cap F \neq \emptyset$ and $(1 \rightarrow (x \rightarrow y)) \cap F \neq \emptyset$ according to the previous lemma. Hence $(1 \rightarrow y) \cap F \neq \emptyset$ by (dHIF3). Therefore, $y \in F$, according to the previous lemma. With this, the validity of the condition (dHF3) is demonstrated and thus the validity of the condition (sHF3). □

Example 3.2. Let $A =: \{a, b, c, 1\}$ be a chain such that

$$\preccurlyeq =: \{(a, a), (a, b), (a, 1), (b, b), (b, 1), (c, c), (c, 1)\}.$$

Let us defile the hyper operations as follows

$\circ$	a	b	c	1
a	{a}	{a}	{a,b,c}	{a}
b	{a}	{a,b}	{b,c}	{a,b}
c	{a,b,c}	{b,c}	{c}	{c}
1	{a}	{a,b}	{c}	{1}

and

$\rightarrow$	a	b	c	1
a	{1}	{1}	{c}	{1}
b	{a.b,c}	{1}	{c}	{1}
c	{a,b}	{a,b}	{1}	{1}
1	{a}	{a,b}	{c}	{1}

Routine calculations show that $(A, \circ, 1, \rightarrow, \preccurlyeq)$ is a hyper (quasi-)ordered residuated system. The subsets $F_1 =: \{1\}$, $F_2 =: \{c, 1\}$, $F_3 =: \{b, 1\}$ and $F_4 =: \{a, b, 1\}$ are implicative strong filters in $\mathfrak{hA}$.

One condition equivalent to condition (sHIF3) in a hyper quasi-ordered residuated system $\mathfrak{hA}$ is described in the following theorem.

Theorem 3.7. *Let F be a non-empty subset of a hyper quasi-ordered residuated system $\mathfrak{hA}$. Then F is an implicative strong filter in $\mathfrak{hA}$ if and only if the subset $F_x = \{u \in A : (x \rightarrow u) \cap F \neq \emptyset\}$ is a strong filter in $\mathfrak{hA}$, for all $x \in A$.*

Proof. ($\Longrightarrow$) Let F be an implicative strong filter in $\mathfrak{hA}$. Then $1 \in F_x$ due to $1 \in F$ and due to the claim (d) in Proposition 2.2. Thus, $F_x \neq \emptyset$.

Let $u, v \in A$ be such that $u \in F_x$ and $u \preccurlyeq v$. Then $(x \rightarrow u) \cap F \neq \emptyset$ and $u \rightarrow v \equiv_{\preccurlyeq} 1 \in F_x$. Thus $(x \rightarrow (u \rightarrow v)) \cap F \neq \emptyset$. Hence $(x \rightarrow v) \cap F \neq \emptyset$ by (sHIF3). This means $v \in F_x$.

Let $u, v \in A$ be such that $u \in F_x$ and $(u \rightarrow v) \cap F_x \neq \emptyset$. Then $(x \rightarrow u) \cap F \neq \emptyset$ and $u \rightarrow v \in F_x$. Thus $(x \rightarrow (u \rightarrow v)) \cap F \neq \emptyset$. So, $(x \rightarrow v) \cap F \neq \emptyset$ by (sHIF3). This means $v \in F_x$.

($\Longleftarrow$) Assume that $F_x = \{u \in A : (x \rightarrow u) \cap F \neq \emptyset\}$ is a strong filter in $\mathfrak{hA}$ for every $x \in A$. Let us prove that F is an implicative strong filter in hA, i.e. prove that F satisfies the conditions (HF2) and (dHIF3).

Let $x, y, z \in A$ be such that $(x \rightarrow y) \cap F \neq \emptyset$ and $(x \rightarrow (y \rightarrow z)) \cap \emptyset$. This means $y \in F_x$ and $y \rightarrow z \in F_x$. Then $y \in F_x$ and $(y \rightarrow z) \cap F_x \neq \emptyset$. Thus, $z \in F_x$ by (dHF3). This shows that F satisfies the condition (dHIF3).

Let $x, y \in A$ be such $x \in F$ and $x \preccurlyeq y$. Then $(1 \rightarrow x) \cap F \neq \emptyset$ by Lemma 3.4(i). This means $x \in F_1$. Thus $y \in F_1$ by (dHF3), that is, $(1 - y) \cap F \neq \emptyset$. From here, again by Lemma 3.4(i), we get that $y \in F$. This shows that F satisfies the condition (HF2). □

The following theorem gives important equivalences for a strong filter in hyper QRS $\mathfrak{hA}$ to be an implicative filter in $\mathfrak{hA}$.

Theorem 3.8. *Let F be a strong filter in a hyper quasi-ordered residuated system $\mathfrak{hA}$. Then the following are equivalent:*

(iii) *F is an implicative strong filter in $\mathfrak{hA}$.*

(iv) $(\forall x, y \in A)((y \rightarrow (y \rightarrow x)) \cap F \neq \emptyset \Longrightarrow (y \rightarrow x) \cap F \neq \emptyset)$.

(v) $((z \rightarrow (y \rightarrow (y \rightarrow x))) \cap F \neq \emptyset \wedge z \in F) \Longrightarrow (y \rightarrow x) \cap F \neq \emptyset$ *for all* $x, y, z \in A$.

Proof. (iii) $\Longrightarrow$ (iv). Let F be an implicative strong filter in $\mathfrak{hA}$ and let $x, y \in A$ be such that $(y \rightarrow (y \rightarrow x)) \cap F \neq \emptyset$. On the one hand, we have that $1 \in F$ due to (HF2). On the other hand, we have $1 \in y \rightarrow y$ by the claim (c) in Proposition 2.2. Hence $(y \rightarrow y) \cap F \neq \emptyset$. From the above, we get $(y \rightarrow x) \cap F \neq \emptyset$ by (dHIF3).

(iv) $\Longrightarrow$ (v). Let (iv) hold and let $x, y, z \in A$ be such that

$$(z \to (y \to (y \to x))) \cap F \neq \emptyset \text{ and } z \in F.$$

Since F is a strong filter in $\mathfrak{hA}$, then $(y \to (y \to x)) \cap F \neq \emptyset$ by (dHF3). Finally, we have $(y \to x) \cap F \neq \emptyset$ by (iv).

(v) $\Longrightarrow$ (iii). Assume that (v) is valid. Let us prove the validity of (dHIF3). For that purpose, let $x, y, z \in A$ be such that $(x \to y) \cap F \neq \emptyset$ and $(x \to (y \to z)) \cap F \neq \emptyset$. Thus, there exists $t \in F$ such that $t \in x - (y - z)$. On the other hand, $t \in x \to (y \to z) \preccurlyeq y \to (x \to z)$ applies to these elements according to claim (l) of Proposition 2.2. This means that there exists $s \in y \to (x \to z)$ such that $t \preccurlyeq s$. Then $s \in F$ by (HF2). Thus, $s \in (y \to x) \to z)) \cap F$. Therefore, $(y \to x) \to z)) \cap F \neq \emptyset$. First, from $s \in y \to (x \to z)$ it follows $s \ll y \to (x \to z)$ due to the reflexivity of the relation $\preccurlyeq$. Then $s \circ y \ll x \to z$ by (H3). From here, we obtain $y \ll s \to (x \to z)$ due to the commutativity of the operation $\circ$ and in accordance with (H3). This means that there exists some $v \in s \to (x \to z)$ such that $y \preccurlyeq v$. Hence, $x \to y \preccurlyeq x \to v$ by claim (p) of Proposition 2.3. This means $x \to y \preccurlyeq x \to (s \to (x \to z))$ and, according to claim 2.2(l), from here we get $x \to y \preccurlyeq s \to (x \to (x \to z))$. Second, since $(x \to y) \cap F \neq \emptyset$, we have $(s \to (x \to (x \to z))) \cap F \neq \emptyset$. In addition, as $s \in F$ is valid, from here it follows $(x \to z) \cap F \neq \emptyset$ according to (v). This shows that F satisfies the condition (dHIF3). $\square$

In what follows, we give an example of a strong filter in a hyper quasi-ordered residuated system.

Example 3.3. Let $A = \langle -\infty, 1] \, (\subseteq \mathbb{R})$. Then A with the natural ordering is a partially ordered set. Let us define the hyper-operations $\circ$ and $\to$ on A as follows:
$(\forall x, y \in A)(x \circ y =: min\{x, y\})$ and
$(\forall x, y \in A)((x \leqslant y \implies x \to y =: 1) \wedge (y < x \implies x \to y =: y))$.
It is not difficult to verify that $\mathfrak{hA} = (A, \circ, 1, \to, \leqslant)$ is a hyper (quasi-)ordered residuated system.

Let's verify that for $a \leqslant 1$, the set of the form $[a, 1]$ is a strong filter in $\mathfrak{hA}$:
- It is obvious that the set F satisfies conditions (HF0) and (HF2).
- First, for $x \in [a, 1]$ and $x \leqslant y$, we immediately have $y \in [a, 1]$. For $x \in [a, 1]$ and $y < x$, from $(x \to y) = y \subseteq [a, 1]$ it follows $y \in [a, 1]$. Therefore, $[a, 1]$ is a filter in $\mathfrak{hA}$.
- Second, for $x \in [a, 1]$ and $x \leqslant y$, we immediately have $y \in [a, 1]$ from $(x \to y) \cap [a, 1] = \{1\} \cap [a, 1] \neq \emptyset$; Also, for $x \in [a, 1]$ and $y < x$, from $(x \to y) \cap [a, 1] = \{y\} \cap [a, 1] \neq \emptyset$ it follows $y \in [a, 1]$. So, the set $[a, 1]$ is a strong filter in $\mathfrak{hA}$.

Let us now show that the set $[a, 1]$ is an implicative strong filter in $\mathfrak{hA}$, verifying condition (iv) in the previous theorem:
- For $x \leqslant y$, we have $y \to x = x$ and $y \to (y \to x) = y \to x = x$ and therefore $(y \to (y \to x)) \cap [a, 1] = (y \cap x) \cap [a, 1] = \{x\} \cap [a, 1] \neq \emptyset$.
- For $y < x$, we have $y \to x = 1$ and $y \to (y \to x) = y \to 1 = 1$ and therefore $(y \to (y \to x)) \cap [a, 1] = (y \to x) \cap [a, 1] = \{1\} \cap [a, 1] \neq \emptyset$.
As condition (iv) is satisfied in both cases, we conclude that $[a, 1]$ is an implicative strong filter in $\mathfrak{hA}$.

However, a strong filter in a hyper quasi-ordered residuated system $\mathfrak{hA}$ does not have to be a strong implicative filter in $\mathfrak{hA}$ as the following example shows:

Example 3.4. Let $A = \{0, a, b, c, 1\}$ and let the order on A be determined as follows $0 < a < b < c < 1$. In this example, we write a instead of $\{a\}$ for simplicity. Let's define hyper operations on A in the following way:

$\circ$	0	a	b	c	1
0	0	0	0	0	0
a	0	{0,a}	{0,a}	{0,a}	a
b	0	{0,a}	{0,a,b}	{a,b}	b
c	0	{0,a}	{a,b}	c	c
1	0	a	b	c	1

and

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	b	1	1	1
c	a	b	b	1	1
1	0	a	b	c	1

Routine calculations show that $(A, \circ, 1, \rightarrow, \preccurlyeq)$ is a hyper (quasi-)ordered residuated system. The subset $F =: \{1\}$ is a strong filter in $\mathfrak{hA}$, but it is not an implicative strong filter in $\mathfrak{hA}$ because, for example, it holds

$$(a \rightarrow (a \rightarrow 0)) \cap \{1\} = (a \rightarrow c) \cap \{1\} = \{1\} \cap \{1\} = \{1\} \neq \emptyset \text{ but}$$
$$(a \rightarrow 0) \cap \{1\} = \{c\} \cap \{1\} = \emptyset.$$

4. Conclusions and future works

The concept of quasi-ordered residuated algebraic structure was introduced in 2018 by Bonzio and Chajda ([1]) as a generalization of both quasi-ordered commutative residuated lattices and hoop-algebras. The filter substructure in this algebraic structure was introduced and developed in [9, 10, 11, 12, 13, 17]. The idea of hyperstructures was applied to these algebraic structures in [18] by designing the concept of hyper quasi-ordered residuated system. In that paper, the concept of a strong filter in a hyper quasi-ordered residuated system was developed.

This article, which is a continuation of research started with paper [18], it is discussed about the implicative strong filter in a hyper quasi-ordered residue system. In addition to defining this concept, some equivalent conditions for its determination are found. Issues not treated in this report are the determination of the concept of implicative filters and its relation to the concept of implicative strong filter in this class of algebraic structures.

Previously mentioned paper and this report may be a substrate for further research on substructures of this class of hyper structures such as comparative, normal and fantastic filters, for example.

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References

[1] S. Bonzio and I. Chajda, Residuated relational systems. Asian-Eur. J. Math., 11(2)(2018), 1850024 doi.org/10.1142/S1793557118500249

[2] R. A. Borzooei and S. Niazian. Weak hyper residuated lattices. Quasigroups Relat. Syst., 21(1)(2013), 29–42.

[3] R. A. Borzooei, M. Bakhshi and O. Zahiri. Filter theory on hyper residuated lattices. Quasigroups Relat. Syst., 22(1)(2014), 33–50.

[4] R. A. Borzooei, H. Varasteh and K. Borna. On hyper hoop-algebras. Ratio Math., 30(2016), 67–81.

[5] R. A. Borzooei and M. Aaly Kologani. Results on hoops. Journal of Algebraic Hyperstructures and Logical Algebras, 1(1)(2020), 61–77. DOI: 10.29252/hatef.jahla.1.1.5

[6] B. Bosbach. Komplementäre halbgruppen. Axiomatik und arithmetik. Fundam. Math., 64 (1969), 257-287.

[7] P. Corsini and V. Leoreanu. Applications of hyperstructure theory. Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[8] F. Marty. Sur une generalization de la notion de groupe. 8th Congress Mathematiciens Scan- dinaves, Stockholm, (1934), 45-49.
[9] D. A. Romano. Filters in residuated relational system ordered under quasi-order. Bull. Int. Math. Virtual Inst., 10(3)(2020), 529–534. DOI: 10.7251/BIMVI2003529R
[10] D. A. Romano. Associated filters in quasi-ordered residuated systems. Contributions to Mathematics, 1(2020), 22–26. DOI: 10.47443/cm.2020.0010
[11] D. A. Romano. Implicative filters in quasi-ordered residuated system. Proyecciones, 40(2)(2021), 417–424. doi: 10.22199/issn.0717-6279-2021-02-0025
[12] D. A. Romano. Comparative filters in quasi-ordered residuated system. Bull. Int. Math. Virtual Inst., 11(1)(2021), 177–184. DOI: 10.7251/BIMVI2101177R
[13] D. A. Romano. Weak implicative filters in quasi-ordered residuated systems. Proyecciones, 40(3)(2021), 797–804. doi: 10.22199/issn.0717-6279-4332
[14] D. A. Romano. Strong quasi-ordered residuated system. Open J. Math. Sci., 5(2021), 73-79. DOI: 10.30538/oms2021.0146
[15] D. A. Romano. Prime and irreducible filters in strong quasi-ordered residuated systems. Open J. Math. Sci., 5(2021), 172–181. doi:10.30538/oms2021.0154
[16] D. A. Romano. Quasi-ordered residuated systems, a review. J. Int. Math. Virtual Inst., 12(1)(2022), 55–86. DOI: 10.7251/JIMVI2201055R
[17] D. A. Romano. Shift filters of quasi-ordered residuated system. Communications in Advanced Mathematical Sciences, 5(3)(2022), 124-130. DOI: 10.33434/cams.1089222
[18] D. A. Romano. Hyper quasi-ordered residuated systems. Mat. Bilten, 46(2)(2022) (In press).
[19] O. Zahiri, R. A. Borzooei and M. Bakhshi. Quotient hyper residuated lattices. Quasigroups Relat. Syst., 20(2012), 125–138.

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