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FUNCTIONS AND IT'S IMPLICATIONS ON ζ-NANO TOPOLOGICAL SPACE

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ABSTRACT. In this paper, we are studying about a function's between domain ζ-nano topological spaces and codomain nano topological spaces which means every nano topology has its inverse image in ζ-nano topology(i.e., ζ-continuous). We establish the ζ-cluster point in ζ-continuous. We search image from a ζ-open sets (ζ-closed set) to a nano open sets(nano closed sets) is called a ζ-open map(ζ-closed map). Finally, some of the results are portrayed with these ζ-continuity with $\mathcal{N}$-continuity and we extended to ζ-homeomorphism.

1. INTRODUCTION AND PRELIMINARIES

Thivagar and Richard [4] found the result of Nano-Topology($\mathcal{N}$). In this paper, we choose ζ element as one of the elements are $\mathcal{L}_{\mathcal{R}}, \mathcal{U}_{\mathcal{R}}$, and $\mathcal{B}_{\mathcal{R}}$. And compare with all open set containing in the nano topological space and then we get some new open elements which is subcollection of ζ element. From this expansion of nano topology was called ζ-nano topology is developed by Jenavee et al [1, 3] and also investigate with some results related with ζ open and ζ closed sets.

Definition 1.1. [4] Let $\mathcal{V}$ be a non-empty finite set of members are called the universe and $\mathcal{R}$ has an equivalence relation on $\mathcal{V}$ known as the indiscernibility relation. Members belonging to the same equivalence class are called to be indiscernible with each other. The pair $(\mathcal{V}, \mathcal{R})$ is called to be the approximation-space. Let $\mathcal{X} \subset \mathcal{V}$.

(1) The lower approximation of $\mathcal{X}$ with respect to $\mathcal{R}$ is the set of all members, which can be for certain classified as $\mathcal{X}$ with respect to $\mathcal{R}$ and it is represented by $\mathcal{L}_{\mathcal{R}}(\mathcal{X})$. That is,
$$\mathcal{L}_{\mathcal{R}}(\mathcal{X}) = \cup_{x \in \mathcal{V}}\{\mathcal{R}(\mathcal{X}) : \mathcal{R}(\mathcal{X}) \subseteq \mathcal{X}\},$$
where $\mathcal{R}(\mathcal{X})$ denoted the equivalence class determined by $\mathcal{X}$.

(2) The upper approximation of $\mathcal{X}$ with respect to $\mathcal{R}$ is the set of all members, which can be possibly classified as $\mathcal{X}$ with respect to $\mathcal{R}$ and it is represented by $\mathcal{U}_{\mathcal{R}}(\mathcal{X})$.
$$\text{(i.e.)}, \mathcal{U}_{\mathcal{R}}(\mathcal{X}) = \cup_{x \in \mathcal{V}}\{\mathcal{R}(\mathcal{X}) : \mathcal{R}(\mathcal{X}) \cap \mathcal{X} \neq \phi\}$$

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(3) The boundary region of $\mathcal{X}$ wit respect to $\mathcal{R}$ is the set of all members, which can be neither in nor as not-$\mathcal{X}$ with respect to $\mathcal{R}$ and it is represented by $\mathcal{B}_{\mathcal{R}}(\mathcal{X})$.

(i.e.), $\mathcal{B}_{\mathcal{R}}(\mathcal{X}) = \mathcal{U}_{\mathcal{R}}(\mathcal{X}) - \mathcal{L}_{\mathcal{R}}(\mathcal{X})$.

Definition 1.2. [4] Let $\mathcal{V}$ be the universe $\mathcal{R}$ be an equivalence relation on $\mathcal{V}$ and $\tau_{\mathcal{R}}(\mathcal{X}) = \{\mathcal{V}, \phi, \mathcal{U}_R(X), \mathcal{L}_{\mathcal{R}}(\mathcal{X}), \mathcal{B}_{\mathcal{R}}(\mathcal{X})\}$, where $\mathcal{X} \subset \mathcal{V}$. Then $\tau_{\mathcal{R}}(\mathcal{X})$ satisfies the following axioms:

(1) $\mathcal{V}$ and $\phi \in \tau_{\mathcal{R}}(\mathcal{X})$.
(2) The union of the members of any sub-collection of $\tau_{\mathcal{R}}(\mathcal{X})$ is in $\tau_{\mathcal{R}}(\mathcal{X})$.
(3) The intersection of the members of finite sub-collection of $\tau_{\mathcal{R}}(\mathcal{X})$ is in $\tau_{\mathcal{R}}(\mathcal{X})$.

That is, $\tau_{\mathcal{R}}(\mathcal{X})$ is a topology on $\mathcal{V}$ is called the Nano topology on $\mathcal{V}$ with respect to $\mathcal{X}$. $(\mathcal{V}, \tau_{\mathcal{R}}(\mathcal{X}))$ is called the Nano topological space. Members of the Nano topology are called Nano open sets in $\mathcal{V}$. Members of $[\tau_{\mathcal{R}}(\mathcal{X})]^c$ are called Nano closed sets.

Definition 1.3. [1] A subset J of a Nano topological space $(V, \mathcal{N}_{\mathcal{R}})$ is called ζ-Nano-open set if there exists a Nano open set $Z \in \mathcal{N}_{\mathcal{R}}\text{-}O$, such that

(1) $Z \neq \phi, V$.
(2) $J \subseteq \mathcal{N}_{\mathcal{R}}\text{-}int(J) \cup Z$.

In $(V, \mathcal{N}_{\mathcal{R}})$, the member of the open set is said to be ζ-Nano-open and the complement is ζ-Nano-closed set. The collection of all ζ-Nano-open including ϕ, V is said to be ζ-Nano-topological space if satisfies topological space definition. So, this $(V, \mathcal{N}_{\mathcal{R}}, \zeta)$ or $\mathcal{N}\text{-}\tau_{\zeta}(J)$ can be rewritten in the form ζ-Nano-topological space on V.

Definition 1.4. [1] Let E be a subset of a ζ-Nano-Topology.

(1) The union of all Nano-ζ sets contained in E is represent in the form of ζ-Nano-int(E). We can rewrite in the form $\zeta_i(E)$.
(2) The intersection of all Nano-ζ sets containing in E is represent in the form of ζ-Nano-cl(E). Also we write in the form $\zeta_c(E)$.
(3) The exterior of ζ-Nano-Topology in E is defined by $\zeta_e(E) = \zeta_i(V - E)$.
(4) The frontier of ζ-Nano-Topology in E is defined by $\zeta_f(E) = \zeta_c(E) \cap \zeta_c(V - E)$.

Proposition 1.1. [1] *In $(V, \mathcal{N}_{\mathcal{R}}, \zeta)$, if E and F are subsets, then the following should be attained.*

(1) $\zeta_{\mathcal{N}}\text{-}i\text{-}(\phi) = \phi$ *and* $\zeta_{\mathcal{N}}\text{-}c\text{-}(\phi) = \phi$.
(2) $\zeta_{\mathcal{N}}\text{-}i\text{-}(V) = V$ *and* $\zeta_{\mathcal{N}}\text{-}c\text{-}(V) = V$.
(3) $\zeta_{\mathcal{N}}\text{-}i\text{-}(E) \subseteq E \subseteq \zeta_{\mathcal{N}}\text{-}c\text{-}(E)$.
(4) $E \subseteq F \Rightarrow \zeta_{\mathcal{N}}\text{-}i\text{-}(E) \subseteq \zeta_{\mathcal{N}}\text{-}i\text{-}(F)$ *and* $\zeta_{\mathcal{N}}\text{-}c\text{-}(E) \subseteq \zeta_{\mathcal{N}}\text{-}i\text{-}(F)$.
(5) $\zeta_{\mathcal{N}}\text{-}i\text{-}(E \cap F) \subseteq \zeta_{\mathcal{N}}\text{-}i\text{-}(E) \cap \zeta_{\mathcal{N}}\text{-}i\text{-}(F)$.
(6) $\zeta_{\mathcal{N}}\text{-}c\text{-}(E \cap F) = \zeta_{\mathcal{N}}\text{-}c\text{-}(E) \cap \zeta_{\mathcal{N}}\text{-}c\text{-}(F)$.
(7) $\zeta_{\mathcal{N}}\text{-}c\text{-}(E \cup F) \supseteq \zeta_{\mathcal{N}}\text{-}c\text{-}(E) \cup \zeta_{\mathcal{N}}\text{-}c\text{-}(F)$.
(8) $\zeta_{\mathcal{N}}\text{-}i\text{-}(E \cup F) = \zeta_{\mathcal{N}}\text{-}i\text{-}(E) \cup \zeta_{\mathcal{N}}\text{-}i\text{-}(F)$.
(9) $\zeta_{\mathcal{N}}\text{-}i\text{-}(\zeta_{\mathcal{N}}\text{-}i\text{-}(E)) \subseteq \zeta_{\mathcal{N}}\text{-}i\text{-}(E)$.
(10) $\zeta_{\mathcal{N}}\text{-}c\text{-}(\zeta_{\mathcal{N}}\text{-}c\text{-}(E)) \supseteq \zeta_{\mathcal{N}}\text{-}c\text{-}(E)$.
(11) $\zeta_{\mathcal{N}}\text{-}i\text{-}(E) = E = \zeta_{\mathcal{N}}\text{-}c\text{-}(E)$.

Theorem 1.2. [3] *In (V, τ_{ζ}), Z is ζ-closed set iff Z = ζ-closure set.*

Corollary 1.3. [3] *In (V, τ_{ζ}), Z is ζ-open set iff Z = ζ-interior set.*

Lemma 1.4. [3] *In (V, τ_{ζ}), $\zeta_c(S)$ is the smallest ζ-closed set containing S.*

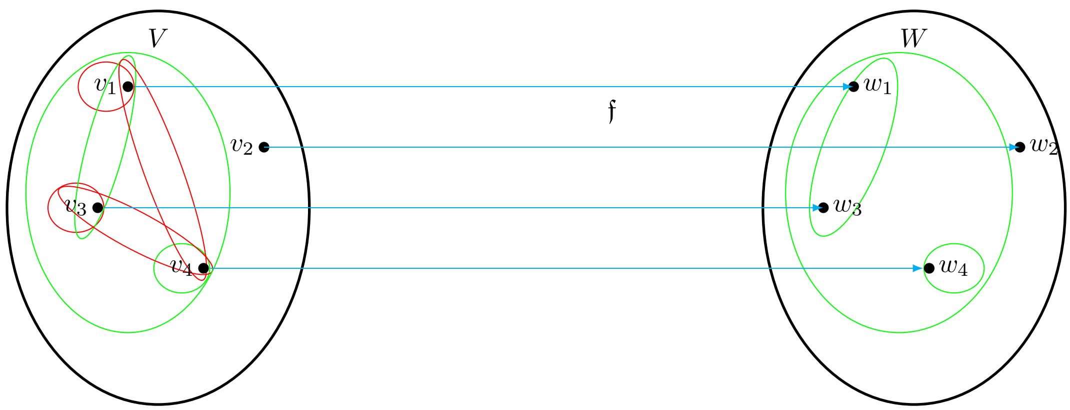
The rest of the paper is content are present: In section-2, gives some functions and it's implications like ζ-continuity, ζ-open map, ζ-closed map, ζ-cluster point and ζ-homeomorphism. Few result is based with these content are present. The conclusion of the current study is set forth in section-3.

2. ζ-Continuity

Definition 2.1. Let $(V, \tau_\zeta(Y))$ is ζ-nano topological space and $(W, \tau_\mathcal{N}(Z))$ is nano topological space. Then a mapping $\mathfrak{f} : (V, \tau_\zeta(Y)) \to (W, \tau_\mathcal{N}(Z))$ is ζ-continuous on V, if $\mathfrak{f}^{-1}$ is a ζ-open set in V for each $\mathcal{N}$-open set Z in W .

Example 2.2. Let $V = \{v_1, v_2, v_3, v_4\}$ with $V \diagup \mathcal{R} = \{\{v_1, v_3\}, \{v_2\}, \{v_4\}\}$. Then, $Y = \{v_1, v_4\} \subseteq V$. So, $\tau_\mathcal{N}(Y) = \{\phi, V, \{v_4\}, \{v_1, v_3, v_4\}, \{v_1, v_3\}\}$. Now, $\zeta_Y = \{v_1, v_3\}$ then $\tau_\zeta(Y) = \{\phi, V, \{v_1\}, \{v_3\}, \{v_4\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_3, v_4\}, \{v_1, v_3, v_4\}\}$. Let $W = \{w_1, w_2, w_3, w_4\}$ with $W \diagup \mathcal{R} = \{\{w_1, w_3\}, \{w_2\}, \{w_4\}\}$. Then, $Z = \{w_1, w_4\}$ $\subseteq W$. So, $\tau_{\acute{\mathcal{N}}}(Z) = \{\phi, W, \{w_4\}, \{w_1, w_3, w_4\}, \{w_1, w_3\}\}$.

Define $\mathfrak{f} : (V, \tau_\zeta(Y)) \to (W, \tau_\mathcal{N}(Z))$ and $\mathfrak{f}$ are denoted as $\mathfrak{f}(v_1) = w_1, f(v_2) = w_2, f(v_3) = w_3, f(v_4) = w_4$. Then, $\mathfrak{f}^{-1}(\{w_4\}) = \{v_4\}, \mathfrak{f}^{-1}(\{w_1, w_3\}) = \{v_1, v_3\}$, $\mathfrak{f}^{-1}(\{w_1, w_3, w_4\}) = \{v_1, v_3, v_4\}$. (i.e) (That is), the inverse image of every ζ-open set in V is $\mathcal{N}$-open in W. Thence, $\mathfrak{f}$ is ζ-continuous.



<u>**Note:**</u>

(1) In codomain, the collection of all $\tau_\mathcal{N}$ open members on W.
(2) In domain, the collection of all τ_ζ open members on V.
(3) In domain, the different colour denotes different open sets,
   (a) red-ζ-nano open sets.
   (b) green-nano open sets.
   (c) cyan-mapping between domain and codomain.

Theorem 2.1. *Every $\mathcal{N}$-continuous function is ζ-continuous.*

Proof. Let $\mathfrak{f} : V \to W$ be a $\mathcal{N}$-continuous function. If Z is $\mathcal{N}$-open in W, then $\mathfrak{f}^{-1}(Z)$ is $\mathcal{N}$-open in V and by definition 1.3, $\mathfrak{f}^{-1}(Z)$ is ζ-open in W. So, using definition 2.1, $\mathfrak{f}$ is ζ-continuous. ☐

Remark. *The converse of the last theorem 2.1 may be false.*

Example 2.3. In example 2.2, $\mathfrak{f}^{-1}(\{v_1, v_4\}) = \{w_1, w_4\}$ is ζ-open but not $\mathcal{N}$-open.

Definition 2.4. Let V be a space and $Z \subseteq V$. Then, a point $z \in V$ is called a ζ-cluster point of Z if for every ζ-open set U containing z, $U \cap (Z - \{z\}) \neq \phi$. .

Proposition 2.2. *In* $(V, \tau_\zeta(Y))$,

1. *If* Z_i *is* ζ*-closed for each* $i \in \mathcal{I}$*, then* $\cap_{i\in\mathcal{I}} Z_i$ *is* ζ*-closed.*
2. *If* Z_i *is* ζ*-open for each* $i \in \mathcal{I}$*, then* $\cup_{i\in\mathcal{I}} Z_i$ *is* ζ*-open.*

Proof. (1) Let that $Z = \cap_{i\in\mathcal{I}} Z_i$ and $z \in \zeta_c(Z)$. By definition 2.4, for every ζ-open set O containing z, $Z \cap O \neq \phi \Rightarrow Z_i \cap O \neq \phi$for each $i \in \mathcal{I}$. If $z \notin Z$, then $z \notin Z_i$ for some $i \in \mathcal{I}$. Since Z_i is ζ-closed, by theorem 1.2, $Z_i = \zeta_c(Z_i)$ and $z \notin \zeta_c(Z_i) \Rightarrow \exists$ a ζ-open set O containing z such that $Z_i \cap O = \phi$. This contradiction gives that $z \in Z$ and $\zeta_c(Z) \subseteq Z$. By using proposition 1.1, it gives $Z = \zeta_c(Z)$. Using theorem 1.2, Z is ζ-closed, (i. e), $\cap_{i\in\mathcal{I}} Z_i$. Hence, it proves (1).

(2) Let Z_i is ζ-open for each $i \in \mathcal{I}$. Then, from the definition 1.3, $V - Z_i$ is ζ-closed for each $i \in \mathcal{I}$. From (i), $\cap_{i\in\mathcal{I}}(V - Z_i)$ is ζ-closed $\Rightarrow V - (\cup_{i\in\mathcal{I}} Z_i)$ is ζ-closed and $\cup_{i\in\mathcal{I}} Z_i$ is ζ-open. Thus, it proves (2).

□

Theorem 2.3. *A function* $\mathfrak{f} : V \to W$ *is* ζ*-continuous* $\Leftrightarrow$ *for each point* $v \in V$ *and each* $\mathcal{N}$*-open set Z in W with* $\mathfrak{f}(v) \in Z$*, there is a* ζ*-open set U in V such that* $v \in U$ *and* $\mathfrak{f}(U) \subseteq Z$.

Proof. Let $\mathfrak{f}$ is ζ-continuous. Assume $v \in V$ and Z be $\mathcal{N}$-open set in W such that $\mathfrak{f}(v) \in Z$. Then, $v \in \mathfrak{f}^{-1}(Z)$ and by definition 2.1, $\mathfrak{f}^{-1}(Z)$ is a ζ-open set in V. Now, take $U = \mathfrak{f}^{-1}(Z)$. Then, U is ζ-open set in V such that $v \in U$ and $\mathfrak{f}(U) \subseteq V$.
$\Leftarrow$, Assume for each point $v \in V$ and each $\mathcal{N}$-open set Z in W with $\mathfrak{f}(v) \in Z$, there is a ζ-open set U in V such that $v \in U$ and $\mathfrak{f}(U) \subseteq Z$. Let Z be $\mathcal{N}$-open in W and $v \in \mathfrak{f}^{-1}(Z)$. Then, $\mathfrak{f}(z) \in Z$. By our assumption, $\exists$ a ζ-open set U_v in V such that $v \in U_v$ and $\mathfrak{f}(U_v) \subseteq Z \Rightarrow v \in U_v \subseteq \mathfrak{f}^{-1}(Z)$ and $\mathfrak{f}^{-1}(Z) = \cup\{U_v : v \in \mathfrak{f}^{-1}(Z)\}$. Using the proposition 2.2, $\mathfrak{f}^{-1}(Z)$ is ζ-open in V and then by definition 2.1, $\mathfrak{f}$ is ζ-continuous. □

Theorem 2.4. *Let* $\mathfrak{f} : V \to W$ *is a function. Then, the following are equivalent:*

1. $\mathfrak{f}$ *is* ζ*-continuous.*
2. *The inverse image of each closed set in W is* ζ*-closed in V.*
3. *For each subset Z of V,* $\mathfrak{f}(\zeta_c(Z)) \subseteq \zeta_c(\mathfrak{f}(Z))$.

Proof. Case (1): $(1) \Rightarrow (2)$. Assume $\mathfrak{f}$ is ζ-continuous. Let P be a $\mathcal{N}$-closed subset of W. Then, $W - P$ is $\mathcal{N}$-open in W. And by definition 2.1, $\mathfrak{f}^{-1}(W - P)$ is ζ-open in V $\Rightarrow V - \mathfrak{f}^{-1}(P)$ is ζ-open in V and $\mathfrak{f}^{-1}(P)$ is ζ-closed in V. Hence, it proves.

Case (2): $(2) \Rightarrow (3)$. Assume the inverse image of every closed set in W is ζ-closed in V. Let $Z \subseteq V$ Then, $\mathfrak{f}(Z) \subseteq \mathcal{N}_c(\mathfrak{f}(Z)) \Rightarrow \mathfrak{f}^{-1}(\mathfrak{f}(Z)) \subseteq \mathfrak{f}^{-1}(\mathcal{N}_c(\mathfrak{f}(Z))) \Rightarrow Z \subseteq \mathfrak{f}^{-1}(\mathcal{N}_c(\mathfrak{f}(Z)))$. Here, $\mathcal{N}_c(\mathfrak{f}(Z))$ is a closed set in W and $\mathfrak{f}^{-1}(\mathcal{N}_c(\mathfrak{f}(Z)))$ is ζ-closed in V containing Z, by our assumption. By lemma 1.4, $Z \subseteq \zeta_c(Z) \subseteq \mathfrak{f}^{-1}(\mathcal{N}_c(\mathfrak{f}(Z))) \Rightarrow \mathfrak{f}(\zeta_c(Z)) \subseteq (\mathfrak{f}(\mathfrak{f}^{-1}(\mathcal{N}_c(\mathfrak{f}(Z))))) \subseteq \mathcal{N}_c(\mathfrak{f}(Z))$. Thus, it proves.

Case (3): $(3) \Rightarrow (1)$. Assume that for each subset Z of V, $\mathfrak{f}(\zeta_c(Z)) \subseteq \mathcal{N}_c(\mathfrak{f}(Z))$. Let P be a $\mathcal{N}$ closed subset of W. Then, $\mathfrak{f}^{-1}(P) \subseteq V$. Since, we have $\mathfrak{f}(\zeta_c(\mathfrak{f}^{-1}(P))) \subseteq \mathcal{N}_c(\mathfrak{f}(\mathfrak{f}^{-1}(Z))) \subseteq \mathcal{N}_c(P) = P \Rightarrow \zeta_c(\mathfrak{f}^{-1}(P)) \subseteq \mathfrak{f}^{-1}(P)$. By theorem 1.1, $\mathfrak{f}^{-1}(P) \subseteq \zeta_c(\mathfrak{f}^{-1}(P))$. Then, $\mathfrak{f}^{-1}(P) = \zeta_c(\mathfrak{f}^{-1}(P))$. It gives, $\mathfrak{f}^{-1}(P)$ is ζ-closed in V. If Q is $\mathcal{N}$-open in W, then W-Q is $\mathcal{N}$-closed in W. Since, $\mathfrak{f}^{-1}(W - Q)$ is ζ-closed in V $\Rightarrow V - \mathfrak{f}^{-1}(Q)$

ζ-closed in V and $\mathfrak{f}^{-1}(Q)$ is ζ-open in V. From definition 2.1, $\mathfrak{f}$ is ζ-continuous. Thence, it proves. □

Corollary 2.5. *A function $\mathfrak{f} : V \to W$ is ζ-continuous $\Leftrightarrow$ for each subset P of W, $\zeta_c(\mathfrak{f}^{-1}(P)) \subseteq \mathfrak{f}^{-1}(\zeta_c(P))$.*

Proof. Let $\mathfrak{f}$ is ζ-continuous and $P \subseteq W$. Then, $\mathfrak{f}^{-1}(P) \subseteq V$. From the theorem 2.4, $\mathfrak{f}(\zeta_c(\mathfrak{f}^{-1}(P))) \subseteq \mathcal{N}_c(\mathfrak{f}(\mathfrak{f}^{-1}(P))) \subseteq \mathcal{N}_c(P)$ and $\zeta_c(\mathfrak{f}^{-1}(P)) \subseteq \mathfrak{f}^{-1}(\mathcal{N}_c(P))$.

$\Leftarrow$(Conversely), let for each subset P of W, $\zeta_c(\mathfrak{f}^{-1}(P)) \subseteq \mathfrak{f}^{-1}(\mathcal{N}_c(P))$ and P be $\mathcal{N}-$open in W. Since, $\zeta_c(\mathfrak{f}^{-1}(P)) \subseteq \mathfrak{f}^{-1}(\mathcal{N}_c(P)) \Rightarrow \mathfrak{f}(\zeta_c(\mathfrak{f}^{-1}(P))) \subseteq \mathfrak{f}(\mathfrak{f}^{-1}(\mathcal{N}_c(P))) \subseteq \mathcal{N}_c(P)$. If $P = \mathfrak{f}(Z)$ where $Z \subseteq V$,, then $\mathfrak{f}(\zeta_c(P)) \subseteq \mathcal{N}_c(\mathfrak{f}(Z))$ and by theorem 2.4, $\mathfrak{f}$ is ζ-continuous. □

Definition 2.5. A function $\mathfrak{f} : V \to W$ is said to be ζ-open map if the image of each open set in V is a ζ-open set in W.
We can rewrite ζ-open map as ζ-open.

Proposition 2.6. *Every $\mathcal{N}$ open map is ζ-open.*

Proof. Let V is $\mathcal{N}$ open. Since, $\exists$ nano open O such that $O = \mathcal{N}_i$. From the definition 1.3, Therefore, V is ζ-open. □

Remark. *The converse of the before theorem 2.6 can be false.*

Example 2.6. Let $V = \{v_1, v_2, v_3, v_4\}$ with $V/\mathcal{R} = \{\{v_1, v_3\}, \{v_2\}, \{v_4\}\}$. Then, $Y = \{v_1, v_4\} \subseteq V$. So, $\tau_{\mathcal{N}}(Y) = \{\phi, V, \{v_4\}, \{v_1, v_3, v_4\}, \{v_1, v_3\}\}$. Now, $\zeta_Y = \{v_1, v_3\}$ then $\tau_\zeta(Y) = \{\phi, V, \{v_1\}, \{v_3\}, \{v_4\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_3, v_4\}, \{v_1, v_3, v_4\}\}$.
Define $\mathfrak{f} : (V, \tau_{\mathcal{N}}(Y)) \to (V, \tau_\zeta(Y))$ and $\mathfrak{f}$ are denoted as $\mathfrak{f}(v_1) = w_1, f(v_2) = w_2, f(v_3) = w_3, f(v_4) = w_4$. $\mathfrak{f}(\{v_3, v_4\}) = \{v_3, v_4\}$ is ζ-open but not $\mathcal{N}$ open.

Theorem 2.7. *A function $\mathfrak{f} : V \to W$ is ζ-open $\Leftrightarrow$ for any subset P of W and any closed subset Q of V containing $\mathfrak{f}^{-1}(P)$, $\exists$ a ζ-closed set R of W containing P such that $\mathfrak{f}^{-1}(R) \subseteq Q$.*

Proof. Let $\mathfrak{f}$ is ζ-open, $P \subseteq W$ and Q be a closed subset of V such that $\mathfrak{f}^{-1}(P) \subseteq Q$. So, V-Q is $\mathcal{N}$-open in V. Since, $\mathfrak{f}$ is ζ-open, by definition 2.5, $\mathfrak{f}(V - Q)$ is ζ-open in W $\Rightarrow W - \mathfrak{f}(V - Q)$ is ζ-closed in W. Since $\mathfrak{f}^{-1}(P) \subseteq Q$, $V - Q \subseteq V - \mathfrak{f}^{-1}(P) = \mathfrak{f}^{-1}(W - P) \Rightarrow \mathfrak{f}(V - Q) \subseteq \mathfrak{f}(\mathfrak{f}^{-1}(W - P)) \subseteq W - P$. So $P \subseteq W - \mathfrak{f}(V - Q)$. Then, $\mathfrak{f}^{-1}(W - \mathfrak{f}(V - Q)) = V - \mathfrak{f}^{-1}(\mathfrak{f}(V - Q)) \subseteq Q$. Hence, $R = W - \mathfrak{f}(V - Q)$ is ζ-closed set of W containing P and $\mathfrak{f}^{-1}(R) \subseteq Q$.

$\Leftarrow$, let reverse part is holds. Let U be $\mathcal{N}$-open in V. Then V-U is $\mathcal{N}$-closed in V and $\mathfrak{f}^{-1}(W - \mathfrak{f}(U)) \subseteq V - U$. Since, $\exists$ a ζ-closed set R of W such that $W - \mathfrak{f}(U) \subseteq R$ and $\mathfrak{f}^{-1}(R) \subseteq V - U$. Therefore, $W - \mathfrak{f}(U) \subseteq R \Rightarrow W - R \subseteq \mathfrak{f}(U)$. And $\mathfrak{f}^{-1}(R) \subseteq V - U \Rightarrow U \subseteq V - \mathfrak{f}^{-1}(R) = \mathfrak{f}^{-1}(W - R) \Rightarrow \mathfrak{f}(U) \subseteq \mathfrak{f}(\mathfrak{f}^{-1}(W - R) \subseteq W - R$. This gives, $\mathfrak{f}(U) = W - R$ and $\mathfrak{f}(U)$ is ζ-open in W. Thus, $\mathfrak{f}$ is ζ-open. □

Theorem 2.8. *If $\mathfrak{f} : V \to W$ is ζ-open, then $\mathfrak{f}(\mathcal{N}_i(Z)) \subseteq \zeta_i(\mathfrak{f}(Z))$ for every subset Z of V.*

Proof. Suppose $\mathfrak{f} : V \to W$ is ζ-open and $Z \subseteq V$. Since, $\mathcal{N}_i(Z)$ is $\mathcal{N}$-open in V, by definition 2.5, $\mathfrak{f}(\mathcal{N}(Z))$ is ζ-open in W and by corollary 1.3, then $\mathfrak{f}(\mathcal{N}_i(Z)) = \zeta_i(\mathfrak{f}(\mathcal{N}_i(Z)))$. So, $\mathfrak{f}(\mathcal{N}(Z)) \subseteq \mathfrak{f}(Z)$. By proposition 1.1(4), $\zeta_i(\mathfrak{f}(\mathcal{N}_i(Z))) \subseteq \zeta_i(\mathfrak{f}(Z))$ and $\mathfrak{f}(\mathcal{N}(Z)) \subseteq \zeta_i(\mathfrak{f}(Z))$. □

Theorem 2.9. *If $\mathfrak{f} : V \to W$ is ζ-open, then $\mathfrak{f}^{-1}(\zeta_c(P)) \subseteq \mathcal{N}_c(\mathfrak{f}^{-1}(P))$ for every subset P of W.*

Proof. Let $\mathfrak{f} : V \to W$ is ζ-open and $P \subseteq Y$. Then, $\mathfrak{f}^{-1}(P) \subseteq \mathcal{N}_c(\mathfrak{f}^{-1}(P))$ and $\mathcal{N}_c(\mathfrak{f}^{-1}(P))$ is closed in V. By theorem 2.7, $\exists$ a ζ-closed set Q of W containing P such that $\mathfrak{f}^{-1}(Q) \subseteq \mathcal{N}_c(\mathfrak{f}^{-1}(P))$. Since, $P \subseteq Q$, by proposition 2.2 and theorem 1.2, $\zeta_c(P) \subseteq \zeta_c(Q) = Q$. So, $\mathfrak{f}^{-1}(\zeta_c(P)) \subseteq \mathfrak{f}^{-1}(Q) \subseteq \mathcal{N}_c(\mathfrak{f}^{-1}(P))$. □

Definition 2.7. A function $\mathfrak{f} : V \to W$ is said to be ζ-closed map if the image of each closed set in V is a ζ-closed set in W. We rewrite ζ-closed map as ζ-closed.

Theorem 2.10. *Every $\mathcal{N}$ closed map is ζ -closed.*

Proof. Let V is $\mathcal{N}$ closed. Since, $\exists$ nano closed C such that $C = \mathcal{N}_c$. From the definition 1.3, Therefore, V is ζ-closed. □

Remark. *The converse of the before theorem 2.10 can be false.*

Example 2.8. In example 2.6, $\mathfrak{f}(\{v_2, v_3\}) = \{v_2, v_3\}$ is ζ-closed but not $\mathcal{N}$-closed.

Theorem 2.11. *A function $\mathfrak{f} : V \to W$ is ζ-closed $\Leftrightarrow$ for each subset P of W and each open set Q in V containing $\mathfrak{f}^{-1}(P)$, $\exists$ a ζ-open set O of W containing P such that $\mathfrak{f}^{-1}(O) \subseteq Q$.*

Proof. Let $\mathfrak{f}$ is ζ-closed and $P \subseteq W$ and Q be an open set of V such that $\mathfrak{f}^{-1}(P) \subseteq Q$. Then, $V - Q$ is closed in V. Since, $\mathfrak{f}$ is ζ-closed, by definition 2.7, $\mathfrak{f}(V - Q)$ is ζ-closed in W. We take $O = W - \mathfrak{f}(V - Q)$. Then O is ζ-open in W. Again since, $\mathfrak{f}^{-1}(P) \subseteq Q$, then $P \subseteq O$ and $\mathfrak{f}^{-1}(O) = \mathfrak{f}^{-1}(W - \mathfrak{f}(V - Q)) = V - \mathfrak{f}^{-1}(\mathfrak{f}(V - Q)) \subseteq Q$.

$\Leftarrow$, let for each subset P of W and each open set Q in V containing $\mathfrak{f}^{-1}(P)$, $\exists$ a ζ-open set O of W containing P such that $\mathfrak{f}^{-1}(O) \subseteq Q$ and G be a closed set of V. Then, V-G is open in V and $\mathfrak{f}^{-1}(W - \mathfrak{f}(G)) \subseteq V - G$. By our assumption, $\exists$ a ζ-open set O of W such that $W - \mathfrak{f}(G) \subseteq O$ and $\mathfrak{f}^{-1}(O) \subseteq V - G$. So, $G \subseteq V - \mathfrak{f}^{-1}(O)$. Then, $W - O \subseteq \mathfrak{f}(G) \subseteq \mathfrak{f}(V - \mathfrak{f}^{-1}(O)) = \mathfrak{f}(\mathfrak{f}^{-1}(W - O) \subseteq W - O \Rightarrow \mathfrak{f}(G) = W - O$. And $\mathfrak{f}(G)$ is ζ-closed in W. This gives $\mathfrak{f}$ is ζ-closed. □

Theorem 2.12. *If $\mathfrak{f} : V \to W$ is ζ-closed, then $\zeta_c(\mathfrak{f}(Z)) \subseteq \mathfrak{f}(\mathcal{N}_c(Z))$ for every subset Z of V.*

Proof. Let $\mathfrak{f} : V \to W$ is ζ-closed and $Z \subseteq V$. Then, $\mathcal{N}_c(Z)$ is closed in V. By definition 2.7, $\mathfrak{f}(\mathcal{N}_c(Z))$ is ζ-closed in W. By theorem 1.2(4),$\zeta_c(\mathfrak{f}(\mathcal{N}_c(Z))) = \mathfrak{f}(\mathcal{N}_c(Z))$. Since, $\mathfrak{f}(Z) \subseteq \mathfrak{f}(\mathcal{N}_c(Z))$, by proposition 1.1(4), $\zeta_c(\mathfrak{f}(Z)) \subseteq \zeta_c(\mathfrak{f}(\mathcal{N}_c(Z))) = \mathfrak{f}(\mathcal{N}_c(Z))$. □

Theorem 2.13. *For each bijective function $\mathfrak{f} : V \to W$, the following are equivalent :*

(1) $\mathfrak{f}^{-1} : W \to V$ is ζ-continuous.
(2) $\mathfrak{f}$ is ζ-open.
(3) $\mathfrak{f}$ is ζ-closed.

Proof. Case(1): Let $\mathfrak{f}^{-1}$ is ζ-continuous. If O is open in V, then by definition 2.1, $(\mathfrak{f}^{-1})^{-1}(O)$ $= \mathfrak{f}(O)$ is ζ-open in W. So, $\mathfrak{f}$ is ζ-open. Hence, $(1) \Rightarrow (2)$.

Case(2): Assume $\mathfrak{f}$ is ζ-open. If C is $\mathcal{N}$-closed set of V, then V C is $\mathcal{N}$-open in V. From definition 2.5, $\mathfrak{f}(V - C) = W - \mathfrak{f}(C)$ is ζ-open in W and so $\mathfrak{f}(C)$ is ζ-closed in W. Thus, $(2) \Rightarrow (3)$.

Case(3): Suppose $\mathfrak{f}$ is ζ-closed. If C is a $\mathcal{N}$-closed set of V, then by definition 2.7, $\mathfrak{f}(C)$ is ζ-closed in W. So, $(\mathfrak{f}^{-1})^{-1}(C)$ is ζ-closed in W. By theorem 2.4, $\mathfrak{f}^{-1}$ is ζ-continuous. Hence, $(3) \Rightarrow (1)$. □

Definition 2.9. A bijection $\mathfrak{f} : V \to W$ is said to be ζ-homeomorphism if both $\mathfrak{f}$ and $\mathfrak{f}^{-1}$ are ζ-continuous.

Theorem 2.14. *Every $\mathcal{N}$-homeomorphism is a ζ-homeomorphism.*

Proof. Suppose, if $\mathfrak{f} : V \to W$ is a $\mathcal{N}$-homeomorphism, then $\mathfrak{f}$ is bijective and both $\mathfrak{f}$ and $\mathfrak{f}^{-1}$ are $\mathcal{N}$-continuous. From the theorem 2.1, $\mathfrak{f}$ and $\mathfrak{f}^{-1}$ are ζ-continuous. So, by definition 2.9, $\mathfrak{f}$ is a ζ-homeomorphism. □

Remark. *The converse before theorem 2.14 can be false.*

Example 2.10. In example 2.6, $\mathfrak{f}(\{v_3, v_4\}) = \{v_3, v_4\}$ is ζ-homeomorphism but not $\mathcal{N}$-homeomorphism because it is not both $\mathfrak{f}$ and $\mathfrak{f}^{-1}$ are ζ-continuous.

Theorem 2.15. *Let $\mathfrak{f} : V \to W$ be a bijective ζ-continuous function. Then, the following are equivalent :*

(1) $\mathfrak{f}$ is ζ-open.
(2) $\mathfrak{f}$ is ζ-homeomorphism.
(3) $\mathfrak{f}$ is ζ-closed.

Proof. Case (1): Assume (1) holds. If O is open in V, then by definition 2.5, $\mathfrak{f}(O)$ is ζ-open in W. But, $\mathfrak{f}(O) = (\mathfrak{f}^{-1})^{-1}(O)$. So,$(\mathfrak{f}^{-1})^{-1}(O)$ is ζ-open in W. From the definition 2.1, $\mathfrak{f}^{-1}$ is ζ-continuous. Thus, (2) is proved.

Case (2): Assume (2) holds. Let C be closed in V. Then, by definition 7.1, $\mathfrak{f}^{-1}$ is ζ-continuous. And by theorem 2.4, $(\mathfrak{f}^{-1})^{-1}(C) = \mathfrak{f}(C)$ is ζ-closed in W. By definition 1.4, $\mathfrak{f}$ is ζ-closed. Thus, (3) is proved.

Case (3): Assume (3) holds. If O is open in V, then V-O is closed in V. By definition 2.7, $\mathfrak{f}(V - O)$ is ζ-closed in W. But $\mathfrak{f}(V - O) = W - \mathfrak{f}(O) \Rightarrow W - \mathfrak{f}(O)$ is ζ-closed in W and so $\mathfrak{f}(O)$ is ζ-open in W. From the definition 1.1, $\mathfrak{f}$ is ζ-open. Thus, (1) is proved. □

3. Conclusion

This paper, we learned the concept of functions like ζ-continuity, ζ-open map, ζ-closed map, ζ-cluster point and ζ-homeomorphism. In the future, we can study in the area of irresolute functions (i.e., the domain and the codomain are the ζ-nano topological spaces) and as well as ζ-open map, ζ-closed map. We can extend by finding an inverse image or image for the ζ closed set to ζ closed set also. And we can learn in various areas of topological spaces with associated applications.

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