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SOME TYPES OF INTERIOR FILTERS IN QUASI-ORDERED Γ-SEMIGROUPS

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ABSTRACT. In this article, the concepts of interior, weak-interior and quasi-interior filters in a quasi-ordered Γ-semigroup are introduced and recognize some of their fundamental properties. In addition to the above, the relationships between these three classes of filters in quasi-ordered Γ-semigroups are considered. One of the specifics in this analysis, among others, is that the requirement that a filter in a quasi-ordered Γ-semigroup S has to be a sub-semigroup in S is omitted. Instead of this requirement in the determination of these three classes of filters the consistency requirement is incorporated.

1. INTRODUCTION

The concept of interior ideals of a semigroup S has been introduced by S. Lajos in [13] as a subsemigroup J of S such that $SJS \subseteq J$. The interior ideals of semigroups have been also studied by G. Szász in [23, 24]. In [6, 7] N. Kehayopulu and M. Tsingelis introduced the concepts of interior ideals in ordered semigroups. W. Jantanan, O. Johdee and N. Praththong in [3] and M. M. Krishna Rao in [9] also wrote about interior ideals in ordered semigroup. Classes of weak-interior ideals and quasi-interior ideals in semigroups were introduced in articles [9, 10] by M. M. Krishna Rao. S. Tarsuslu and G. Çuvalcıoğlu in [25] also dealt with quasi-interior ideals in semigroups. These types of ideals in quasi-ordered semigroups were analyzed in [18] by D. A. Romano.

The notion of a Γ-semigroup was introduced in 1984 in [20] by M. K. Sen. Then many authors took part in the development of this concept as well as many of its properties and substructures (see, for example [1, 2, 16, 21, 22]). Classes of weak-interior ideals and quasi-interior ideals in Γ-semigroup were introduced in articles [9, 11] by M. M. Krishna Rao. Thus Y. B. Jun and S. Lajos ([5]), Y. I. Kwon ([12]), S. K. Lee and S. S. Lee ([14]), S. K. Lee and Y. I. Kwon ([15]), K. Hila ([1]), A. Iampan ([2]), N. Kehayopulu and M. Tsingelis ([8]) and Jyothi V. et al. ([4]) analyzed the filters in quasi-ordered Γ-semigroups.

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As we know, although many results on semigroups (about ordered semigroups) can be transferred into Γ-semigroups (res. into po-Γ-semigroups) just putting a symbol Γ in the appropriate place, while for some other results for the transfer needs subsequent technical changes ([8], page 97), we decide to look at these phenomena by considering some special filter substructures in quasi-ordered Γ-semigroups.

The paper [19] discusses the design of interior, weak-interior and quasi-interior filters in quasi-ordered semigroups.

In this article, the concepts of interior (Subsection 3.1), weak-interior (Subsection 3.2) and quasi-interior (Subsection 3.3) filters in a quasi-ordered Γ-semigroup are introduced and analyze of their fundamental properties are discussed. For the purposes of this report, the requirement that the mentioned filters in a quasi-ordered Γ-semigroup S be subsemigroups of S has been omitted. Instead, the consistency requirement is incorporated into the determination of these filters. The chosen orientation allows that the families of these filters form complete lattices. In addition to the above, the relationships between these three classes of filters in quasi-ordered Γ-semigroups are considered. Thus, for example, some satisfactory conditions were found that filters and interior filters coincide in a quasi-ordered Γ-semigroup. Also, if a quasi-ordered Γ-semigroup S satisfies one additional condition, then filters, interior filters and weak interior filters of S are coincide.

2. Preliminaries

Let S be a set. A relation $\preceq \subseteq S \times S$ is a quasi-order on S if holds

(1) $(\forall x \in S)(x \preceq x)$,

(2) $(\forall x, y, z \in S)((x \preceq y \wedge y \preceq z) \implies x \preceq z)$.

If the quasi-order relation is present on the set S, then we say that the set S is quasi-ordered. A quasi-order relation $\leqslant$ on a set S is a partial order on S if the following holds

(3) $(\forall x, y \in S)((x \leqslant y \wedge y \leqslant x) \implies x = y)$.

In this case, for the set S is said to be an ordered set, or, in short, to be a po-set. A quasi-order in a set S does not have to be an order in S, in the general case. If S is a semigroup with respect to the internal binary operation '$\cdot$', then the quasi-order $\preceq$ must be compatible with the operation in the following sense

(4) $(\forall x, y, u \in S)(x \preceq y \implies (ux \preceq uy \wedge xy \preceq yu))$.

Let S and Γ be two non-empty sets. S is called a Γ-semigroup if there exist mapping from $S \times \Gamma \times S$ to S, written as $(x, a, y) \longmapsto xay$ satisfying the identity

(5) $(\forall x, y, z \in S)(\forall a, b \in \Gamma)((xay)bz = xa(ybc))$.

It is known that the concept of Γ-semigroups is a generalization of the notion of semigroups. A Γ-semigroup S is called a quasi-ordered Γ-semigroup if holds

(6) $(\forall x, y, u \in S)(\forall a \in \Gamma)(x \preceq y \implies (xau \preceq yau \wedge uax \preceq uay))$.

A sufficient number of examples of ordered Γ-semigroups can be found in the literature (see, for example, [1]).

Example 2.1. Let $S =: M_{2\times 2}$ be a semigroup of real matrices of type 2×2 over the field $\mathbb{R}$ of real numbers and $\Gamma = M_{2\times 2}$. The ternary operation in S over $\mathbb{R}$ is the standard multiplication of matrices. Then $M_{2\times 2}$ is an ordered Γ-semigroup under the relation $\leqslant$ defined by

$$(\forall A, B \in M_{2\times 2})(A \leqslant B \iff (\forall\, ij \in \{1, 2\})(A_{ij} \leqslant B_{ij})).$$

Let S be a quasi-ordered Γ-semigroup. By a sub-semigroup of S we mean a non-empty subset A of S such that

(7) $(\forall x, y \in S)(\forall a \in \Gamma)((x \in A \wedge y \in A) \implies xay \in A)$.

A non-empty subset J of a quasi-ordered Γ-semigroup S is called a right ideal of S if $J\Gamma S \subseteq J$ and the following holds

(8) $(\forall u, v \in S)((v \in J \wedge u \preceq v) \implies u \in J)$.

A non-empty subset J of a Γ-semigroup S is called a left ideal of S if (8) and $S\Gamma J \subseteq J$ holds. A subset J is called an ideal of S if it is both a left and a right ideal of S. It is obvious that a (left, right) ideal in a quasi-ordered Γ-semigroup S is a sub-semigroup in S.

A subset F of a quasi-ordered Γ-semigroup S is said to be a right filter of S if the following holds

(9) $(\forall x, y \in S)(\forall \alpha \in \Gamma)(x\alpha y \in F \implies y \in F)$ and

(10) $(\forall x, y \in S)((x \in F \wedge x \preceq y) \implies y \in F)$.

A subset F of a quasi-ordered Γ-semigroup S is said to be a left filter of S if (10) holds and the following implication is valid

(11) $(\forall x, y \in S)(\forall \alpha \in \Gamma)(x\alpha y \in F \implies x \in F)$.

A subset F of a quasi-ordered Γ-semigroup S is said to be a (two side) filter of S if it is both a left filter and a right filter of S.

Remark. Our determination of (left, right) filters in a quasi-ordered Γ-semigroup S here differs from the determination of the concept of filters in the texts [1, 8, 15, 17]: We omit the requirement that (left, right) filter should be a sub-semigroup of S since there is no such condition in the determination of ideals in such semigroups. In addition, it is also not required that a filter be a non-empty subset of Γ-semigroup S. In each individual case (F is an interior filter, a weak-interior filter or a quasi-interior filter, which will be discussed in the next section), we will comment on the determination of the filter with additional condition

(A) $(\forall x, y \in S)(\forall a \in \Gamma)((x \in F \wedge y \in F) \implies xay \in F)$

and what it produces in these special cases.

Of course, it could be said that the substructure in a quasi-ordered Γ-semigroup determined on this way is a generalized (left, right) filter, and then, for simplicity of writing, omit the adjective 'generalized'.

3. The main results

The material presented in this section is a central part of this paper. In this section, we introduce the concepts of interior filters, weak-interior filters and quasi-interior filters in a quasi-ordered Γ-semigroup and analyze their basic properties. In addition, some sufficient conditions have been found that connect these three classes of filters in such Γ-semigroups.

3.1. **Interior filters.** The concept of interior ideal of a semigroup S has been introduced by S. Lajos in [13] as a sub-semigroup J of a semigroup S such that $SJS \subseteq J$. The interior ideals of semigroups have been also studied by G. Szász in [23, 24]. Let S be a quasi-ordered Γ-semigroup. A non-empty sub-semigroup J of S is called an interior ideal of S if (8) and the following $S\Gamma J\Gamma S \subseteq J$ holds. So, a subset J of a quasi-ordered Γ-semigroup S is an interior ideal of S if the following are valid:

(a) $J \neq \emptyset$,

(7) $(\forall x, y \in S)(\forall a \in \Gamma)((x \in J \wedge y \in J) \implies xay \in J)$,

(12) $(\forall x, u, v \in S)(\forall a, b \in \Gamma)(x \in J \implies uaxbv \in J)$ and
(8) $(\forall u, v \in S)((v \in J \wedge u \preceq v) \implies u \in J)$.

In the following definition we create the concept of interior filters of a quasi-ordered Γ-semigroup as a dual of the concept of ordered interior ideals in such a semigroup.

Definition 3.1. A subset F of a quasi-ordered Γ-semigroup S is an interior filter of S if (10) is valid and the following holds
(13) $(\forall x, y \in S)(\forall a \in \Gamma)(xay \in F \implies (x \in F \vee y \in F))$,
(14) $(\forall u, v, x \in S)(\forall a, b \in \Gamma)(uaxbv \in F \implies x \in F)$.

Remark. It should be noted that an interior filter F in a quasi-ordered Γ-semigroup S does not have to be either a non-empty subset of S or a sub-semigroup of S. Instead, the consistency requirement (13) is incorporated into the determination of this filter. Thus, the $\emptyset$ and S are trivial filters in a quasi-ordered Γ-semigroup S However, in the analysis that follows, we will not omit to analyze this substructure when it additionally satisfies condition (A).

Without much difficulty, by direct verification the following proposition can be shown to be valid:

Proposition 3.1. *Let J be an interior ideal of a quasi-ordered Γ-semigroup S. Then, the set $S \setminus J$ is an interior filter of S. If the ideal J of S satisfies the condition*
(P) $(\forall x, y \in S)(\forall a \in \Gamma)(xay \in J \implies (x \in J \vee y \in J))$,
then the filter $S \setminus J$ satisfies the condition (A).

Proof. Let J be an interior ideal of a quasi-ordered Γ-semigroup S. This means that J satisfies the conditions (a), (7), (12) and (8). Put $F =: S \setminus J$. Since the contraposition of (7) gives (13) and the contraposition of (12) gives (14), it remains to prove (10).

Let $x, y \in S$ be such that $x \in S \setminus J$ and $x \preceq y$. If it were $y \in J$, we would have $x \in J$ by (8), which is contrary to the assumption $x \notin J$. So it has to be $y \in S \setminus J$.

If the ideal J satisfies condition (P), then the statement (A) holds for the filter F since (A) is a contraposition of (P). □

Example 3.2. Let $S = \{0, 1, 2, 3, 4\}$ and '$\cdot$' defined on S as follows:

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	3	2
3	0	3	3	3	3
4	0	4	2	3	4

Put $\Gamma = S$. A mapping $S \times \Gamma \times S \longrightarrow S$ is defined as $xay =$ usual product of $x, y, a \in S$. The quasi-order relation on S is given by

$$\preceq = \{(0,0), (0,1), (0,2), (0,3), (0,4), (1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,4)\}.$$

Then S forms a quasi-ordered Γ-semigroup. By direct verification one can establish that the sets $\{1, 2, 3, 4\}$, $\{1, 2, 4\}$ and $\{1\}$ are interior filters in Γ-semigroup S.

The following theorem connects the concept of filters and the concept of interior filters in a quasi-ordered Γ-semigroup.

Theorem 3.2. *Any filter in a quasi-ordered Γ-semigroup S is an interior filter in S.*

Proof. Let F be a filter in a quasi-ordered Γ-semigroup S. This means that F satisfies the conditions (9), (10) and (11). Let us prove the condition (13) and (14).

Let $x, y \in S$ and $a \in \Gamma$ be such that $xay \in F$. Then $x \in F$ and $y \in F$ by (9) and (11) because F is a right and left filter in S.

Let $u, v, x \in S$ and $a, b \in \Gamma$ be such that $uaxbv \in F$. Then $uax \in F$ by (11) since F is a right filter in S. Also, from $uax \in F$ it follows $x \in F$ by (9) since F is a left filter in S. This proves that F satisfies the condition (14).

So, F is an interior filter in S. □

The inverse of the previous theorem is not valid as the following example shows:

Example 3.3. Let $S = \{a, b, c, d\}$ and operations '$\cdot$' defined on S as follows:

$\cdot$	a	b	c	d
a	a	c	d	d
b	c	d	d	d
c	b	d	d	d
d	d	d	d	d

Put $\Gamma = S$. A mapping $S \times \Gamma \times S \longrightarrow S$ is defined as $xay =$ usual product of $x, y, a \in S$ and $\preceq =$: $\{(a, a), (a, b), (a, c), (b, b), (c, c), (d, d)\}$. Then S forms a quasi-ordered Γ-semigroup. By direct verification one can establish that the sets $\{a, b\}$ and $\{a, b, c\}$ are interior filters in Γ-semigroup S but they are neither left filters nor right filters in S.

The inverse of the Theorem 3.2 can be proved in one special case: A quasi-ordered Γ-semigroup S is called intra-regular ([8]) if holds

(IR) $(\forall x \in S)(\forall a \in \Gamma)(\exists u, v \in S)(\exists b, c \in \Gamma)(x \preceq ubxaxcv)$.

It is called left (right) regular if holds

(LR) $(\forall x \in S)(\forall a \in \Gamma)(\exists b \in \Gamma)(\exists u \in S)(x \preceq ubxax)$

((RR) $(\forall x \in S)(\forall a \in \Gamma)(\exists b \in \Gamma)(\exists v \in S)(x \preceq xaxbv)$, res.)

Theorem 3.3. *Let S be an intra-regular quasi-ordered Γ-semigroup. Then any interior filter in S is a filter in S.*

Proof. Let F be an interior filter in S. This means that F satisfies conditions (10), (13) and (14). (9) and (11) need to be proved.

Let $x, y \in S$ and $a \in \Gamma$ be such that $xay \in F$. Then for any $b \in \Gamma$ there exist elements $u, v \in S$ and $c, d \in \Gamma$ such that $xay \preceq uc(xay)b(xay)dv$ by (IR). Thus $uc(xay)b(xay)dv \in F$ by (10). On the one hand, we have that $(ucx)ayb(xaydv) \in F$ gives $y \in F$ by (14), while, on the other hand, we have that $(ucxay)bxa(ydv) \in F$ gives $x \in F$ by (14). Therefore, F is a filter in S. □

We can, also, show that if S is a right (left) regular quasi-order Γ-semigroup, then any interior filter in S is a left (right) filter in S.

Theorem 3.4. *Let a quasi-ordered Γ-semigroup S is a right regular. Then any interior filter in S is a left filter in S.*

Proof. Let F be an interior filter in a quasi-ordered right regular Γ-semigroup S. The validity of formula (11) needs to be proven.

Let $x, y \in S$ and $a \in \Gamma$ be such that $xay \in F$. Then for any $b \in \Gamma$ there exist the elements $u \in S$ and $c \in \Gamma$ such that $xay \preceq (xay)b(xay)cv$ according (RR). Thus $(xay)bxa(ycv) \in F$ by (10). Hence $x \in F$ according to (14). □

However, since in [8], Proposition 6, it is shown that if an ordered Γ-semigroup S is a left (right) regular, then S is intra-regular, the previous theorem immediately follows from Theorem 3.3 if $\preceq$ is an order relation.

If we replace the condition (RR) in a quasi-ordered semigroup S with the condition

(B) $(\forall x, v \in S)(\forall a \in \Gamma)(x \preceq xav)$

the validity of Theorem 3.4 will be preserved.

Theorem 3.5. *Let S be a quasi-ordered Γ-semigroup that satisfies the condition (B). Then any interior filter in S is a left filter in S.*

Proof. If a quasi-ordered Γ-semigroup S satisfies condition (B), then it is a right regular Γ-semigroup. Indeed, from $(\forall x, z \in S)(\forall a \in \Gamma)(x \preceq xaz)$ follows $(\forall x, v \in S)(\forall a, b \in \Gamma)(x \preceq xa(xbv) = (xax)bv)$ which means that S is a right regular Γ-semigroup. Hence, F is a left filter in S according Theorem 3.4. □

The condition (B) can be weakened.

Theorem 3.6. *Let S be a quasi-ordered Γ-semigroup that satisfies the condition*

(C) $(\forall x \in S)(\forall a \in \Gamma)(x \preceq xax)$.

Then any interior filter in S is a left filter in S.

Proof. Let S be a quasi-ordered Γ-semigroup that satisfies the condition (C) and let F be an interior filter in S. If $x, y \in S$ and $a, b \in \Gamma$ are elements such that $xby \in F$, then $xaxby \in F$ by (10) because $xby \preceq (xax)by$ follows from $x \preceq xax$ by (6). Thus $x \in F$ by (14). □

Our next theorem connects the terms 'interior ideal' and 'interior filter' in a quasi-ordered Γ-semigroup S.

Theorem 3.7. *If F $(\neq S)$ is an interior filter of a quasi-ordered Γ-semigroup S, then the set F^c is an interior ideal of S. In addition to the previous one, if F satisfies the condition (A), then the ideal F^c satisfies the condition (P).*

Proof. It should be shown that the set F^c satisfies the following conditions (a), (7), (12) and (8).

The condition $F \neq S$ ensures that the set F^c is inhabited.

Let $x, y \in S$ and $a \in \Gamma$ be arbitrary elements such that $x \notin F$ and $y \notin F$. Then $xay \notin F$ or $xay \in F$. The second option $xay \in F$ would give $x \in F \vee y \in F$ by (13), which contradicts the assumptions. Therefore, it must be $xay \notin F$.

Let $x, u, v \in S$ and $a, b \in \Gamma$ be arbitrary elements such that $x \notin F$. Then $uaxbv \notin$ or $uaxbv \in F$. The second option $uaxbv \in F$ would give $x \in F$ by (14), which contradicts the assumption. Therefore, it must be $uaxbv \notin F$.

Let $u, v \in S$ be such that $u \preceq v$ and $v \in F^c$. If $u \in F$, we would have $v \in F$ according to (10), which is contrary to the assumption $v \notin F$. So, it has to be $u \in F^c$. □

The family $\mathfrak{Intf}(S)$ of all internal filters of a quasi-ordered Γ-semigroup S is not empty because $S \in \mathfrak{Intf}(S)$ and $\emptyset \in \mathfrak{Intf}(S)$. Additionally, the following applies:

Theorem 3.8. *The family $\mathfrak{Intf}(S)$ of all interior filters of a quasi-ordered Γ-semigroup S forms a complete lattice.*

Proof. Let $\{F_i\}_{i \in I}$ be a non-empty family of interior filters of a quasi-ordered Γ-semigroup S.

(a) Let $x, y \in S$ and $a \in \Gamma$ be such that $xay \in \bigcup_{i \in I} F_i$. Then there exists an index $k \in I$ such that $xay \in F_k$. Thus $x \in F_k \subseteq \bigcup_{i \in I} F_i$ or $y \in F_k \subseteq \bigcup_{i \in I} F_i$ by (13). This means that the set $\bigcup_{i \in I} F_i$ satisfies the condition (13).

Let $u, v, x \in S$ and $a \in \Gamma$ be such that $uaxbv \in \bigcup_{i \in I} F_i$. Then there exists an index $k \in I$ such that $uaxbv \in F_k$. Thus $x \in F_k \subseteq \bigcup_{i \in I} F_i$ by (14). This means that the set $\bigcup_{i \in I} F_i$ satisfies the condition (14).

Let $u, v \in S$ be such that $u \in \bigcup_{i \in I} F_i$ and $u \preceq v$. Then there exists an index $k \in I$ such that $u \in F_k$. Thus $v \in F_k \subseteq \bigcup_{i \in I} F_i$.

We conclude that the set $\bigcup_{i \in I} F_i$ is an interior filter of S.

(b) Let X be the family of all interior filters of the quasi-ordered Γ-semigroup S contained in $\bigcap_{i \in I} F_i$. Then $\cup X$ is the maximal interior filter of S contained in $\bigcap_{i \in I} f_i$, according to (a).

(c) If we put $\sqcup_{i \in I} f_i = \bigcup_{i \in I} F_i$ and $\sqcap_{i \in I} F_i = \cup X$, then $(\mathfrak{Intf}(S), \sqcup, \sqcap)$ a is a complete lattice. □

The previous theorem supports our commitment expressed in the determination of the concept of filters in quasi-ordered Γ-semigroups: The union of sub-semigroups $\{F_i\}_{i \in I}$ of S does not have to be a sub-semigroup of S, in the general case.

Corollary 3.9. *For any subset X of a quasi-ordered Γ-semigroup S there is the maximal interior filter contained in X.*

Proof. The proof of this Corollary is obtained directly from part (b) of the evidence in the previous theorem. □

Corollary 3.10. *For any element $x \in S$ there is the maximal interior filter F_x in a quasi-ordered Γ-semigroup S such that $x \notin F_x$.*

Proof. One should take $X = \{u \in S : u \neq x\}$ and apply the previous corollary. □

3.2. **Weak-interior filters.** This subsection is devoted to designing the concept of weak-interior filters of a quasi-ordered Γ-semigroup and recognizing its fundamental properties. The concept of weak interior ideals in Γ-semigroup was introduced in [11] as follows:

- A non-empty subset J of a Γ-semigroup S is said to be left weak interior ideal of S if J is a Γ-sub-semigroup of S and $S\Gamma J \Gamma J \subseteq J$. The last inclusion should be understood in the following sense

(15) $(\forall u, x, y \in S)(\forall a, b \in \Gamma)((x \in J \wedge y \in J) \implies uaxby \in J)$.

-A non-empty subset J of a Γ-semigroup S is said to be right weak-interior ideal of S if J is a Γ-sub-semigroup of S and $J\Gamma J\Gamma S \subseteq J$. The last inclusion should be understood in the following sense

(16) $(\forall x, y, v \in S)(\forall a, b \in \Gamma)((x \in J \wedge y \in J) \implies xaybv \in J)$.

- A non-empty subset J of a Γ-semigroup S is said to be weak interior ideal of S if J is a left weak interior ideal and a right weak interior ideal of S.

A weak interior ideal of a Γ-semigroup S need not be an interior ideal of Γ-semigroup S (see, for example, [11], Remark 3.1 and Example 3.1).

Dual of concept of (left, right) weak interior ideals in a quasi-ordered Γ-semigroup is created in the following way:

Definition 3.4. Let F be a subset of a quasi-ordered Γ-semigroup S.

- F is a left weak interior filter of S if (10) and (13) are valid and the following holds

(17) $(\forall x, y, u \in S)(\forall a, b \in \Gamma)(uaxby \in F \implies (x \in F \vee y \in F))$;

- F is a right weak interior filter of S if (10) and (13) are valid and the following holds
(18) $(\forall x, y, v \in S)(\forall a, b \in \Gamma)(xaybv \in K \implies (x \in K \vee y \in K))$;

- F is a weak interior filter of S if F is a left weak interior filter and a right weak interior filter of S.

Remark. The requirement for a weak-interior filter to be a sub-semigroup of a quasi-ordered Γ-semigroup is omitted similarly as in the case of interior filter determination.

Proposition 3.11. *Let J be a left weak interior ideal of a quasi-ordered Γ-semigroup S. Then, the set $S \setminus J$ is a left weak interior filter of S. If the left weak interior ideal J of S satisfies the condition (P), then the weak interior filter $S \setminus J$ satisfies the condition (A).*

Proof. Let J be a left weak interior ideal of a quasi-ordered Γ-semigroup S. This means that J satisfies the conditions (a), (7), (8) and (15). (10), (13) and (17) need to be proved.

(13) is the contraposition of (7) so it is a valid formula. (17) is the contraposition of (15). Thus, (17) is a valid formula. (10) is obtained from (8) as shown in Proposition 3.1. □

Example 3.5. Let $\mathbb{Q}$ be a field of rational numbers, $S := \{\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \mid b, d \in \mathbb{Q}\}$ be the semigroup of matrices over the filed $\mathbb{Q}$ and $\Gamma = S$. The ternary operation in S over $\mathbb{Q}$ is the standard multiplication of matrices. Then S is an ordered Γ-semigroup. Then $F =: \{\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \mid d \in \mathbb{Q} \wedge d \neq 0\}$ is a left weak interior filter of the Γ-semigroup S and F is neither a left filter nor a right filter, not a weak interior filter and not an interior filter of the Γ-semigroup S.

The following theorem proves that the concept of left weak interior filter in a quasi-ordered Γ-semigroup is well determined.

Theorem 3.12. *Let F $(\neq S)$ be a left weak interior filter of a Γ-semigroup with apartness S. The the set F^c is a left weak interior ideal in S. In addition to the previous one, if F satisfies the condition (A), then the ideal F^c satisfies the condition (P).*

Proof. Let F be a left weak interior filter in a quasi-ordered Γ-semigroup. This means that F satisfies conditions (10), (13) and (17). It should be shown that the set F^c satisfies the following conditions (a), (7), (15) and (8).

The condition $F \neq S$ ensures that the set F^c is inhabited. So, $F^c \neq \emptyset$.

Let $a \in \Gamma$ and $x, y \in S$ be arbitrary elements such that $x \notin F$ and $y \notin F$. Then $xay \in F$ or $xay \notin F$. The first option would give $x \in F$ or $y \in F$ by (13) which is contrary to the assumptions $x \notin F$ and $y \notin F$. Therefore, it must be $xay \notin F$. This means that F satisfies the condition (7).

Let $a, b \in \Gamma$ and $u, x, y \in S$ be such that $x \notin F$ and $y \notin F$. Then $uaxby \in F$ or $uaxby \notin F$. The first option would give $x \in F$ or $y \in F$ by (17) which is contrary to the assumptions $x \notin F$ and $y \notin F$. So, it has to be $uaxby \in F^c$. □

The connection between the concept of right filters and the concept of left weak interior filters in a quasi-ordered Γ-semigroup is described in the following theorem.

Theorem 3.13. *Every right filter of a quasi-ordered Γ-semigroup S is a left weak interior filter of S.*

Proof. Let F be a right filter of a quasi-ordered Γ-semigroup S. This means that (9) and (10) are valid formulas. It should be proved that (10), (13) and (17) are valid formulas. Since (9) or (11) implies (13), we only need to prove (17).

Let $u, x, y \in S$ and $a, b \in \Gamma$ such that $(uxa)by = uaxby \in F$. Then $y \in F$ because F is a right filter of S. So, F is a left weak interior filter of S. □

Analogous to the previous one, it can be proved:

Theorem 3.14. *Any left filter of a quasi-ordered Γ-semigroup is a right weak interior filter in S.*

Therefore:

Theorem 3.15. *Any filter in a quasi-ordered Γ-semigroup S is a weak interior filter in S.*

The following theorem can be considered as the inverse of the Theorem 3.13.

Theorem 3.16. *Let a quasi-ordered Γ-semigroup S satisfy the condition (C). Then any left weak interior filter in S is a right filter in S.*

Proof. Let F be a left weak interior filter in S. This means that (10), (13) and (17) are valid. (10) and (11) need to be proved.

Let $x, y \in S$ and $a \in \Gamma$ be arbitrary element such that $xay \in F$. Since S satisfies the condition (C), we have $y \preceq yby$ for any $b \in \Gamma$. Then $xay \preceq xa(yby)$ by (6). Thus $xayby \in F$ by (10). Hence $y \in F$ by (17). This means that F is a right filter in S. □

Analogous to the previous one, it can be proved:

Theorem 3.17. *Any right weak interior filter in a quasi-ordered Γ-semigroup S is a left filter in S, if S satisfies the condition (C).*

Therefore:

Theorem 3.18. *If a quasi-ordered Γ-semigroup S satisfies the condition (C), then any weak interior filter in S is a filter in S.*

The relationship between interior filter and weak interior filter in a quasi-ordered Γ-semigroup is described by the following theorem.

Theorem 3.19. *Every interior filter of a quasi-ordered Γ-semigroup S is a left weak interior filter of S.*

Proof. Let F be an interior filter of a quasi-ordered Γ-semigroup S. This means that F satisfies (10), (13) and (14). It only needs to be proven (17). Let $u, x, y \in S$ and $a, b \in \Gamma$ be arbitrary elements such that $uaxby \in F$. Then $x \in F$ because F is an interior filter of S. Thus $x \in F \vee y \in F$ which means that F is a left weak interior filter in S. □

Of course, analogous to the previous one, it can be shown:

Theorem 3.20. *Any interior filter in a quasi-ordered Γ-semigroup S is a right weak-interior filter in S.*

Therefore:

Theorem 3.21. *Any interior filter in a quasi-ordered Γ-semigroup S is a weak interior filter in S.*

Combining Theorem 3.18 and Theorem 3.15, we obtain the inverse of Theorem 3.21

Theorem 3.22. *Let a quasi-ordered Γ-semigroup S satisfy the condition (C). Then any weak interior filter in S is an interior filter in S.*

The family $\mathfrak{W}_{\mathfrak{l}}\mathfrak{Intf}(S)$ of all left weak interior filters of a quasi-ordered Γ-semigroup S is not empty because $S \in \mathfrak{W}_{\mathfrak{l}}\mathfrak{Intf}(S)$ and $\emptyset \in \mathfrak{W}_{\mathfrak{l}}\mathfrak{Intf}(S)$. Actually:

Theorem 3.23. *The family $\mathfrak{W}_{\mathfrak{l}}\mathfrak{Intf}(S)$ of all left weak interior filters of a Γ-semigroup S forms a complete lattice.*

Proof. Let $\{F_i\}_{i\in I}$ be a family of left weak interior filters of a quasi-ordered Γ-semigroup S.

(a) Let $x, y \in S$ and $a \in \Gamma$ be such that $xay \in \bigcup_{i\in I} F_i$. Then there exists an index $k \in I$ such that $xay \in F_k$. Thus $x \in F_k \subseteq \bigcup_{i\in I} K_i$ or $y \in F_k \subseteq \bigcup_{i\in I} F_i$ by (13). This means that the set $\bigcup_{i\in I} F_i$ satisfies the condition (13).

Let $u, x, y \in S$ and $a, b \in \Gamma$ be such that $uaxby \in \bigcup_{i\in I} F_i$. Then there exists an index $k \in I$ such that $uaxby \in F_k$. Thus $x \in F_k \subseteq \bigcup_{i\in I} F_i$ or $y \in F_k \subseteq \bigcup_{i\in I} F_i$ by (17). This shows that the set $\bigcup_{i\in I} F_i$ satisfies the condition (17).

Let $x, y \in S$ be such that $x \preceq y$ and $x \in \bigcup_{i\in I} F_i$. Then there exists an index $k \in I$ such that $x \in F_k$. Thus $y \in F_k \subseteq \bigcup_{i\in I} F_i$ by (10). So, the set $\bigcup_{i\in I} F_i$ satisfies the condition (10).

Hence, we conclude that the set $\bigcup_{i\in I} F_i$ is a left weak interior filter of S.

(b) Let X be the family of all left weak interior filters of Γ-semigroup S contained in $\bigcap_{i\in I} F_i$. Then $\cup X$ is the maximal left weak interior filter of S contained in $\bigcap_{i\in I} F_i$, according to (a).

(c) If we put $\sqcup_{i\in I} F_i = \bigcup_{i\in I} F_i$ and $\sqcap_{i\in I} F_i = \cup X$, then $(\mathfrak{W}_{\mathfrak{l}}\mathfrak{Intf}(S), \sqcup, \sqcap)$ a is a complete lattice. □

Corollary 3.24. *For any subset X of a quasi-ordered Γ-semigroup S there is the maximal left weak interior filter contained in X.*

Proof. The proof of this Corollary is obtained directly from part (b) of the evidence in the previous theorem. □

Corollary 3.25. *For any element $x \in S$ there is the maximal left weak interior filter F_x in a quasi-ordered Γ-semigroup S such that $x \notin F_x$.*

Proof. One should take $X = \{u \in S : u \neq x\}$ and apply the previous corollary. □

Without major difficulties, the previous claims concerning the left weak interior filters can be transformed into the claims concerning the right weak interior filters.

3.3. **Quasi-interior filters.** In this subsection, firstly, we will recall the determination of the notions of left, right, and two-sided quasi-interior ideals in a quasi-ordered Γ-semigroups.

- A non-empty subset J of a quasi-ordered Γ-semigroup S is said to be left quasi-interior ideal of S, if J is a Γ-sub-semigroup of S and (8) and $S\Gamma J\Gamma S\Gamma J \subseteq J$ are hold. Thus, the following formulas

(a) $J \neq \emptyset$,
(7) $(\forall x, y \in S)(\forall a \in \Gamma)((x \in J \wedge y \in J) \implies xay \in J)$,
(8) $(\forall x, yS)((x \preceq y \wedge y \in J) \implies x \in J)$, and
(19) $(\forall u, v, x, y \in S)(\forall a, b, c \in \Gamma)((y \in J \wedge u \in J) \implies uaxbvcy \in J)$

are valid formulas in a Γ-semigroup S.

- A non-empty subset J of a quasi-ordered Γ-semigroup S is said to be right quasi-interior ideal of S, if J is a Γ-sub-semigroup of S and (8) and $J\Gamma S\Gamma J\Gamma S \subseteq J$ are hold. Thus, the following formulas

(a) $J \neq \emptyset$,

(7) $(\forall x, y \in S)(\forall a \in \Gamma)((x \in J \wedge y \in J) \implies xay \in J)$,

(8) $(\forall x, yS)((x \preceq y \wedge y \in J) \implies x \in J)$, and

(20) $(\forall x, y, u, v \in S)(\forall a, b, c \in \Gamma)((x \in J \wedge y \in J) \implies xaubycv \in J)$

are valid formulas in S.

- A non-empty subset J of a quasi-ordered Γ-semigroup S is said to be quasi-interior ideal of S, if J is a left quasi-interior ideal and a right quasi-interior ideal of S.

In this subsection, we introduce the notion of left (right) quasi-interior filters as a generalization of ordered interior filters of a quasi-ordered Γ-semigroup and study its properties.

Definition 3.6. Let S be a quasi-ordered Γ-semigroup.

- A subset F of S is said to be left quasi-interior filter of S if (10) and (13) are valid and the following holds

(21) $(\forall u, v, x, y \in S)(\forall a, b, c \in \Gamma)(uaxbvcy \in K \implies (x \in F \vee y \in F))$,

- A subset F of S is said to be right quasi-interior filter of S if (10) and (13) are valid and the following holds

(22) $(\forall x, y, u, vS)(\forall a, b, c \in \Gamma)(xaubycv \in K \implies (x \in F \vee y \in F))$,

- A subset F of S is said to be quasi-interior filter of S if it is both a left quasi-interior filter and a right quasi-interior filter of S.

Remark. As in the previous two cases of interior filters and weak interior filters in quasi-ordered Γ-semigroups here as well, in determining the concept of quasi-interior filters of a quasi-ordered Γ-semigroup S, we omit the requirement that this filter will be a sub-semigroup of S. However, we will not avoid considering this class of filters if they meet this additional condition.

Example 3.7. Let $\mathbb{Q}$ be a field of rational numbers, $S := \{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Q}\}$ be the semigroup of matrices over the filed $\mathbb{Q}$ and $\Gamma = S$. The ternary operation in S over $\mathbb{Q}$ is the standard multiplication of matrices. Then S is an ordered Γ-semigroup. Then $F =: \{\begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} \mid d \in \mathbb{Q} \wedge d \neq 0\}$ is a right quasi-interior filter of S.

Theorem 3.26. *Let S be a quasi-ordered Γ-semigroup. If $F\ (\neq S)$ is a left quasi-interior filter of S, then the set F^c is a left quasi-interior ideal of S. In addition to the previous one, if F satisfies the condition (A), then the ideal F^c satisfies the condition (P).*

Proof. That the set F^c is non-empty follows from the condition $F \neq S$.

Let $x, y \in S$ and $a \in \Gamma$ be arbitrary element such that $x \notin F$ and $y \notin F$. Then $xay \in F$ or $xay \notin F$. The first option would give $x \in F$ or $y \in F$ by (13) which is contrary to assumptions. Therefore, it must be $xay \notin F$. This means $xay \in F^c$.

Let $x, y \in S$ be such that $x \preceq y$ and $y \in F^c$. Suppose $x \in F$. Then it would be $y \in F$ by (10) which is contrary to the hypothesis $y \notin F$. So it has to be $x \notin F$. This proves the validity of formula (8).

Let $x, y, u, v \in S$ and $a, b, c \in \Gamma$ be arbitrary elements such that $x \notin F$ and $y \notin F$. Then $uaxbycv \in F$ or $xaybzcu \notin F$. The first option $uaxbycv \in F$ would give $x \in F$ or $y \in F$ by (21) because F is a left quasi-interior gilter of S, which contradicts the hypotheses $\notin F$ and yF. So, it must be $xaybzcu \notin F$. These prove that (19) holds for F^c. □

Analogous to the previous, it can be proved:

Theorem 3.27. *Let S be a quasi-ordered Γ-semigroup. If F $(\neq S)$ is a right quasi-interior filter of S, then the set F^c is a right quasi-interior ideal of S.*

The notion of quasi-interior filters is a generalization of the notion of filters in quasi-ordered Γ-semigroups as shown by the following theorem.

Theorem 3.28. *Every interior filter of a quasi-ordered Γ-semigroup S is a left quasi-interior filter of S.*

Proof. Let F ne an interior filter of a S. This means that F satisfies the conditions (10), (13) and (14). That F is a left quasi-interior filter is sufficient to prove (21).

Let $x, y, u, v \in S$ and $a, b, c \in \Gamma$ be such that $uaxb(vcy) = uaxbvcy \in F$. Then $x \in F$ by (14) because $vcy \in S$ and F is an interior filter of S. Thus $x \in F \vee y \in F$. So, the set F is a left quasi interior filter of S. □

The reverse of the previous theorem can be proved if the quasi-ordered Γ-semigroup S satisfies one additional condition.

Theorem 3.29. *Suppose that a quasi-ordered Γ-semigroup S satisfies one additional condition:*

(B) *For every elements $x, v \in S$ and $a \in \Gamma$ the following holds $x \preceq xav$.*

Then the interior filters and the left quasi-interior filters in S coincide.

Proof. Suppose that a quasi-ordered Γ-semigroup S satisfies the condition (C) and let F be a left quasi-interior filter in S. This means that F satisfies the conditions (10), (13) and (21). It needs to be proven (14).

Let $x, u, v \in S$ and $a, b, c \in \Gamma$ be arbitrary elements such that $uaxbv \in F$. On the other hand, we have $uaxbv \preceq uaxbvcx$ according to (B). Hence $uaxbvcx\ inF$ by (10). Thus $x \in F$ according to (21). So, the set F is an interior filter in S. □

Since any filter in a quasi-ordered Γ-semigroup S is an interior filter in S, according to Theorem 3.2, immediately from the Theorem 3.28 it follows:

Corollary 3.30. *Every filter of a quasi-ordered Γ-semigroup S is a left quasi-interior filter of S.*

Theorem 3.31. *The family $\mathfrak{Q}_{\mathfrak{l}}\mathfrak{intf}(S)$ of all left quasi-interior filters of a quasi-ordered Γ-semigroup S forms a complete lattice.*

Proof. Let $\{F_i\}_{i \in I}$ be a family of left quasi-interior filters of a quasi-ordered Γ-semigroup S.

(a) Let $x, y \in S$ and $a \in \Gamma$ be arbitrary elements such that $xay \in \bigcup_{i \in I} F_i$. Then there exists an index $k \in I$ such that $xay \in F_k$. Thus $x \in F_k \subseteq \bigcup_{i \in I} F_i$ or $y \in F_k \subseteq \bigcup_{i \in I} F_i$ by (13). This means that the set $\bigcup_{i \in I} F_i$ satisfies the condition (13).

Let $x, y \in S$ be such that $x \preceq y$ and $x \in \bigcup_{i \in I} F_i$. Then there exists an index $k \in I$ such that $x \in F_k$. Thus $y \in F_k \subseteq \bigcup_{i \in I} F_i$ by (10). So, the set $\bigcup_{i \in I} F_i$ satisfies the condition (10).

Let $x, y, u, v \in S$ and $a, b, c \in \Gamma$ be arbitrary elements such that $uaxbvcy \in \bigcup_{i \in I} F_i$. Then there exists an index $l \in I$ such that $uaxbvcy \in F_k$. Thus $x \in F_k \subseteq \bigcup_{i \in I} F_i$ or $y \in F_k \subseteq \bigcup_{i \in I} f_i$ because F_k is a left quasi-interior filter of S.

Hence, $\bigcup_{i \in I} F_i$ is a left quasi-interior filter of S.

(b) Let X be the family of all left quasi-interior filters contained in $\bigcap_{i\in I} F_i$. Then $\cup X$ is the maximal left quasi-interior filter contained in X, according to (a) in this proof.

(c) If we put $\sqcup_{i\in I} F_i = \bigcup_{i\in I} F_i$ and $\sqcap_{i\in I} F_i = \cup X$, then $(\mathfrak{Q}_{\mathfrak{l}}\mathfrak{intf}(S), \sqcup, \sqcap)$ is a complete lattice. □

Analogous to the previous, it can be proved:

Theorem 3.32. *The family $\mathfrak{Q}_{\mathfrak{r}}\mathfrak{intf}(S)$ of all right quasi-interior filters of a quasi-ordered Γ-semigroup S forms a complete lattice.*

4. Conclusions

Various types of filters in ordered $Gamma$ -semigroups are the subject of studies by several authors in the second decade of this century. One of the main specifics of determining the concept of filter F in an ordered Γ-semigroup S is that the filter F must be Γ-subsemigroup of S. In this text, the concepts of interior Γ-filters, weak interior Γ-filters and quasi-interior Γ-filters in a quasi-ordered Γ-semigroup are introduced and analyzed. The main specificity in these determinations is the omission of the requirement that these filters have to be Γ-subsemigroups of the observed Γ-semigroup. Speaking in formal language, it can be said that the substructures of a quasi-ordered Γ-semigroup designed in this way are generalized filters. Of course, the question of the role of these classes of substructures in the theory of ordered Γ-semigroups arises quite naturally. Apart from the reasons for research provided by the logical possibility of their existence, can concepts designed in this way participate in further and deeper analysis of the structure of the ordered Γ-semigroup in which they will play a crucial role?

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