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# POSITIVE SOLUTIONS FOR NONLINEAR CAPUTO-HADAMARD FRACTIONAL RELAXATION DIFFERENTIAL EQUATIONS

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ABSTRACT. We study the existence and uniqueness of positive solutions of the nonlinear fractional relaxation differential equation

$$\begin{cases} \mathfrak{D}_1^{\alpha} x(t) + wx(t) = f(t, x(t)), \ 1 < t \leq e, \\ x(1) = x_0 > 0, \end{cases}$$

where $\mathfrak{D}_1^{\alpha}$ is the Caputo-Hadamard fractional derivative of order $0 < \alpha \leq 1$. In the process we convert the given fractional differential equation into an equivalent integral equation. Then we construct an appropriate mapping and employ the Schauder fixed point theorem and the method of upper and lower solutions to show the existence of a positive solution of this equation. We also use the Banach fixed point theorem to show the existence of a unique positive solution. Finally, an example is given to illustrate our results.

## 1. INTRODUCTION

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[12], [15] and the references therein.

Zhang in [15] investigated the existence and uniqueness of positive solutions for the nonlinear fractional differential equation

$$\begin{cases} D^{\alpha} x(t) = f(t, x(t)), \ 0 < t \leq 1, \\ x(0) = 0, \end{cases}$$

where $D^{\alpha}$ is the standard Riemann Liouville fractional derivative of order $0 < \alpha < 1$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function. By using the method of the upper and lower solution and cone fixed-point theorem, the author obtained the existence and uniqueness of a positive solution.

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The nonlinear fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x\left(t\right)=f(t,x(t))+{}^{C}D^{\alpha-1}g\left(t,x\left(t\right)\right),\ 0<t\leq T, \\ x\left(0\right)=\theta_{1}>0,\ x'\left(0\right)=\theta_{2}>0, \end{cases}$$

has been investigated in [7], where ${}^{C}D^{\alpha}$ is the standard Caputo's fractional derivative of order $1<\alpha\leq 2$, $g,f:[0,T]\times[0,\infty)\to[0,\infty)$ are given continuous functions, $g$ is non-decreasing on $x$ and $\theta_{2}\geq g\left(0,\theta_{1}\right)$. By employing the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the authors obtained positivity results.

In [8], Chidouh, Guezane-Lakoud and Bebbouchi discussed the existence and uniqueness of the positive solution of the following nonlinear fractional relaxation differential equation

$$\begin{cases} {}^{C}D^{\alpha}x\left(t\right)+wx\left(t\right)=f\left(t,x\left(t\right)\right),\ 0<t\leq 1, \\ x\left(0\right)=x_{0}>0, \end{cases}$$

where $0<\alpha\leq 1$, $w>0$ and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is a given continuous function. By using the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the existence and uniqueness of solutions have been established.

Ahmad and Ntouyas in [1] studied the existence and uniqueness of solutions to the following boundary value problem

$$\begin{cases} \mathfrak{D}_{1}^{\alpha}\left(\mathfrak{D}_{1}^{\beta}u(t)-g\left(t,u_{t}\right)\right)=f(t,u_{t}),\ t\in[1,b], \\ u(t)=\phi(t),\ t\in[1-r,1], \\ \mathfrak{D}_{1}^{\beta}u(1)=\eta\in\mathbb{R}, \end{cases}$$

where $\mathfrak{D}_{1}^{\alpha}$ and $\mathfrak{D}_{1}^{\beta}$ are the Caputo-Hadamard fractional derivatives, $0<\alpha,\beta<1$. By employing the fixed point theorems, the authors obtained existence and uniqueness results.

In this paper, we extend the results in [8] by proving the positivity of solutions for the nonlinear fractional relaxation differential equation

$$\begin{cases} \mathfrak{D}_{1}^{\alpha}x\left(t\right)+wx\left(t\right)=f\left(t,x\left(t\right)\right),\ 1<t\leq e, \\ x\left(1\right)=x_{0}>0, \end{cases} \tag{1.1}$$

where $\mathfrak{D}_{1}^{\alpha}$ is the Caputo-Hadamard fractional derivative of order $0<\alpha\leq 1$, $w>0$, $f:[1,e]\times[0,\infty)\to[0,\infty)$ is a given continuous function. To show the existence and uniqueness of the positive solution, we transform (1.1) into an integral equation and then by the method of upper and lower solutions and use the Schauder and Banach fixed point theorems.

This paper is organized as follows. In section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later sections. Also, we present the inversion of (1.1) and the Banach and Schauder fixed point theorems. For details on the Banach and Schauder theorems we refer the reader to [14]. In Sections 3 and 4, we give and prove our main results on positivity and we provide an example to illustrate our results.

## 2. Preliminaries

Let $X=C\left([1,e]\right)$ be the Banach space of all real-valued continuous functions defined on the compact interval $[1,e]$, endowed with the maximum norm. Define the cone

$$\mathcal{E}=\left\{x\in X:x(t)\geq 0,\ \forall t\in[1,e]\right\}.$$

We introduce some necessary definitions, lemmas and theorems which will be used in this paper. For more details, see [10].

**Definition 2.1** ([10]). The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $x : [1, +\infty) \to \mathbb{R}$ is defined as

$$\mathfrak{I}_1^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s) \frac{ds}{s},\ \alpha > 0.$$

**Definition 2.2** ([10]). *T*he Caputo-Hadamard fractional derivative of order $\alpha > 0$ for a continuous function $x : [1, +\infty) \to \mathbb{R}$ is defined as

$$\mathfrak{D}_1^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n(x)(s) \frac{ds}{s},\ n-1 < \alpha < n,$$

where $\delta^n = \left(t\frac{d}{dt}\right)^n$, $n \in \mathbb{N}$.

**Lemma 2.1** ([10]). *Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and $x \in C^n([1,T])$. Then*

$$\left(\mathfrak{I}_1^\alpha \mathfrak{D}_1^\alpha x\right)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(1)}{\Gamma(k+1)} (\log t)^k.$$

**Lemma 2.2** ([10]). *For all $\mu > 0$ and $\nu > -1$,*

$$\frac{1}{\Gamma(\mu)} \int_1^t \left(\log \frac{t}{s}\right)^{\mu-1} (\log s)^\nu \frac{ds}{s} = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} (\log t)^{\mu+\nu}.$$

**Definition 2.3** ([13]). The two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},\ \alpha > 0,\ \beta \in \mathbb{C},\ z \in \mathbb{C}.$$

For $\beta = 1$, we obtain the Mittag-Leffler function in one parameter

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},\ \alpha > 0,\ z \in \mathbb{C}.$$

**Lemma 2.3** ([13]). *The generalized Mittag-Leffler function $E_{\alpha,\beta}(-x)$ with $x \geq 0$ is completely monotonic if and only if $0 < \alpha \leq 1$ and $\beta \geq \alpha$. In other words, it yields*

$$(-1)^n \frac{d^n}{dx^n} E_{\alpha,\beta}(-x) \geq 0 \text{ for all } n \in \mathbb{N}.$$

*Obviously,* $0 \leq E_{\alpha,\beta}(-x) \leq \frac{1}{\Gamma(\beta)}$ *where* $x \geq 0$, $0 \leq \alpha \leq 1$ *and* $\beta \geq \alpha$.

The following lemma is fundamental to our results.

**Lemma 2.4.** *Let $x \in C([1,e])$, $x'$ exists, then $x$ is a solution of (1.1) if and only if*

$$\begin{aligned} x(t) &= x_0 E_\alpha\left(-w(\log t)^\alpha\right) \\ &+ \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) f(s, x(s)) \frac{ds}{s},\ 1 \leq t \leq e. \end{aligned} \tag{2.1}$$

*Proof.* It is easy to prove by the Laplace transform. □

Lastly in this section, we state the fixed point theorems which enable us to prove the existence and uniqueness of a positive solution of (1.1).

**Definition 2.4.** Let $(X, \|.\|)$ be a Banach space and $\mathcal{A} : X \to X$. The operator $\mathcal{A}$ is a contraction operator if there is an $\lambda \in (0, 1)$ such that $x, y \in X$ imply

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \lambda \|x - y\|.$$

**Theorem 2.5** (Banach [14])**.** *Let $\mathcal{K}$ be a nonempty closed convex subset of a Banach space $X$ and $\mathcal{A} : \mathcal{K} \to \mathcal{K}$ be a contraction operator. Then there is a unique $x \in \mathcal{K}$ with $\mathcal{A}x = x$.*

**Theorem 2.6** (Schauder [14])**.** *Let $\mathcal{K}$ be a nonempty bounded, closed and convex subset of a Banach space $X$ and $\mathcal{A} : \mathcal{K} \to \mathcal{K}$ be a completely continuous operator. Then $\mathcal{A}$ has a fixed point in $\mathcal{K}$.*

## 3. Existence of positive solutions

In this section, we consider the results of existence problem for many cases of (1.1). We express (2.1) as

$$x(t) = (\mathcal{A}x)(t), \tag{3.1}$$

where the operators $\mathcal{A} : \mathcal{E} \to X$ are defined by

$$\begin{aligned}(\mathcal{A}x)(t) &= x_0 E_\alpha\left(-w(\log t)^\alpha\right) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) f(s, x(s)) \frac{ds}{s}.\end{aligned}$$

We need the following lemmas to establish our results.

**Lemma 3.1.** *Assume that $f : [1, e] \times [0, \infty) \to [0, \infty)$ is continuous. Then, the operator $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is completely continuous.*

*Proof.* By Lemma 2.3 and taking into account that $f$ is continuous nonnegative function, we get that $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is continuous. The function $f : [1, e] \times B_\eta \to [0, \infty)$ is bounded, then there exists $\rho > 0$ such that

$$0 \leq f(t, x(t)) \leq \rho,$$

where $B_\eta = \{x \in \mathcal{E} : \|x\| \leq \eta\}$. We obtain

$$\begin{aligned}|(\mathcal{A}x)(t)| &= \Big| x_0 E_\alpha\left(-w(\log t)^\alpha\right) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) f(s, x(s)) \frac{ds}{s}\Big| \\ &\leq x_0 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, x(s))| \frac{ds}{s} \\ &\leq x_0 + \frac{\rho}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \leq x_0 + \frac{\rho(\log t)^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

Thus,

$$\|\mathcal{A}x\| \leq x_0 + \frac{\rho}{\Gamma(\alpha+1)}.$$

Hence, $\mathcal{A}(B_\eta)$ is uniformly bounded.

Now, we will prove that $\mathcal{A}(B_\eta)$ is equicontinuous. Let $x \in B_\eta$, then for any $t_1, t_2 \in [1,e]$, $t_2 > t_1$, we have

$$\begin{aligned}
&\left|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)\right| \\
&= \Big| x_0 E_\alpha\left(-w(\log t_2)^\alpha\right) - x_0 E_\alpha\left(-w(\log t_1)^\alpha\right) \\
&+ \int_1^{t_2} \left(\log\frac{t_2}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log\frac{t_2}{s}\right)^\alpha\right) f(s,x(s))\frac{ds}{s} \\
&- \int_1^{t_1} \left(\log\frac{t_1}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log\frac{t_1}{s}\right)^\alpha\right) f(s,x(s))\frac{ds}{s}\Big| \\
&\le x_0\left|E_\alpha\left(-w(\log t_2)^\alpha\right) - E_\alpha\left(-w(\log t_1)^\alpha\right)\right| \\
&+ \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left|\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right| |f(s,x(s))|\frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1} |f(s,x(s))|\frac{ds}{s} \\
&\le x_0\left|E_\alpha\left(-w(\log t_2)^\alpha\right) - E_\alpha\left(-w(\log t_1)^\alpha\right)\right| \\
&+ \frac{\rho}{\Gamma(\alpha)}\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha-1} - \left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s} + \frac{\rho}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&\le x_0\left|E_\alpha\left(-w(\log t_2)^\alpha\right) - E_\alpha\left(-w(\log t_1)^\alpha\right)\right| \\
&+ \frac{\rho}{\Gamma(\alpha+1)}\left((\log t_1)^\alpha + \left(\log\frac{t_2}{t_1}\right)^\alpha - (\log t_2)^\alpha + \left(\log\frac{t_2}{t_1}\right)^\alpha\right) \\
&\le x_0\left|E_\alpha\left(-w(\log t_2)^\alpha\right) - E_\alpha\left(-w(\log t_1)^\alpha\right)\right| + \frac{2\rho}{\Gamma(\alpha+1)}\left(\log\frac{t_2}{t_1}\right)^\alpha.
\end{aligned}$$

As $t_1 \to t_2$ and taking into account that the function $E_\alpha\left(-w(\log t)^\alpha\right)$ is continuous on $[1,e]$, the right-hand side of the previous inequality is independent of $x$ and tends to zero. Thus that $\mathcal{A}(B_\eta)$ is equicontinuous. So, the compactness of $\mathcal{A}$ follows by Ascoli Arzela's theorem. □

Now for any $x \in [a,b] \subset \mathbb{R}^+$, we define respectively the upper and lower control functions as follows

$$H(t,x) = \sup_{a\le y\le x} f(t,y),\ h(t,x) = \inf_{x\le y\le b} f(t,y).$$

It is clear that these functions are nondecreasing on $[a,b]$.

**Definition 3.1.** Let $\overline{x}, \underline{x} \in \mathcal{E}$, $a \le \underline{x} \le \overline{x} \le b$, satisfying

$$\begin{aligned}
&\overline{x}(t) \ge x_0 E_\alpha\left(-w(\log t)^\alpha\right) \\
&\quad + \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log\frac{t}{s}\right)^\alpha\right) H(s,\overline{x}(s))\frac{ds}{s},\ 1 \le t \le e,
\end{aligned}$$

and

$$\begin{aligned}
&\underline{x}(t) \le x_0 E_\alpha\left(-w(\log t)^\alpha\right) \\
&\quad + \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log\frac{t}{s}\right)^\alpha\right) h(s,\underline{x}(s))\frac{ds}{s},\ 1 \le t \le e.
\end{aligned}$$

Then the functions $\overline{x}$ and $\underline{x}$ are called a pair of upper and lower solutions for the equation (1.1).

**Theorem 3.2.** *Assume that $f : [1, e] \times [0, \infty) \to [0, \infty)$ is continuous, and $\overline{x}$ and $\underline{x}$ are respectively upper and lower solutions of (1.1), then the problem (1.1) has at least one positive solution.*

*Proof.* Let

$$\mathcal{K} = \{x \in \mathcal{E},\ \underline{x}(t) \leq x(t) \leq \overline{x}(t),\ t \in [1, e]\}.$$

As $\mathcal{K} \subset \mathcal{E}$ and $\mathcal{K}$ is a nonempty bounded, closed and convex subset. By Lemma 3.1, $\mathcal{A} : \mathcal{K} \to \mathcal{E}$ is completely continuous. Next, we show that if $x \in \mathcal{K}$, we have $\mathcal{A}x \in \mathcal{K}$. For any $x \in \mathcal{K}$, we have $\underline{x} \leq x \leq \overline{x}$, then

$$\begin{aligned}(\mathcal{A}x)(t) &= x_0 E_\alpha\left(-w(\log t)^\alpha\right) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) f(s, x(s)) \frac{ds}{s} \\ &\leq x_0 E_\alpha\left(-w(\log t)^\alpha\right) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) H(s, \overline{x}(s)) \frac{ds}{s} \\ &\leq \overline{x}(t),\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}(\mathcal{A}x)(t) &= x_0 E_\alpha\left(-w(\log t)^\alpha\right) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) f(s, x(s))\, ds \\ &\geq x_0 E_\alpha\left(-w(\log t)^\alpha\right) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) h(s, \underline{x}(s))\, ds \\ &\geq \underline{x}(t).\end{aligned} \tag{3.3}$$

Thus, from (3.2) and (3.3), we obtain that $\mathcal{A}x \in \mathcal{K}$. We now see that all the conditions of the Schauder fixed point theorem are satisfied. Thus there exists a fixed point $x$ in $\mathcal{K}$. Therefore, the problem (1.1) has at least one positive solution $x$ in $\mathcal{K}$. □

**Corollary 3.3.** *Assume that $f : [1, e] \times [0, \infty) \to [0, \infty)$ is continuous, and there exist $\lambda_1, \lambda_2 \geq 0$ such that*

$$\lambda_1 \leq f(t, x) \leq \lambda_2,\ (t, x) \in [1, e] \times [0, +\infty). \tag{3.4}$$

*Then the problem (1.1) has at least one positive solution $x \in \mathcal{E}$, moreover*

$$x(t) \geq x_0 E_\alpha\left(-w(\log t)^\alpha\right) + \lambda_1 (\log t)^\alpha E_{\alpha,\alpha+1}\left(-w(\log t)^\alpha\right), \tag{3.5}$$

*and*

$$x(t) \leq x_0 E_\alpha\left(-w(\log t)^\alpha\right) + \lambda_2 (\log t)^\alpha E_{\alpha,\alpha+1}\left(-w(\log t)^\alpha\right). \tag{3.6}$$

*Proof.* From (3.4) and the definition of control functions, we have

$$\lambda_1 \leq h(t, x) \leq H(t, x) \leq \lambda_2. \tag{3.7}$$

Now, let

$$\mathfrak{D}_1^\alpha \overline{x}(t) + w\overline{x}(t) = \lambda_2,\ \overline{x}(1) = x_0. \tag{3.8}$$

The above equation (3.8) has a positive solution

$$\begin{aligned}\overline{x}(t) &= x_0 E_\alpha\left(-w(\log t)^\alpha\right) + \lambda_2 \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right)\frac{ds}{s} \\ &= x_0 E_\alpha\left(-w(\log t)^\alpha\right) + \lambda_2 (\log t)^\alpha E_{\alpha,\alpha+1}\left(-w(\log t)^\alpha\right).\end{aligned}$$

Taking into account (3.7), we have

$$\begin{aligned}\overline{x}(t) &= x_0 E_\alpha\left(-w(\log t)^\alpha\right) + \lambda_2 \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right)\frac{ds}{s} \\ &\geq x_0 E_\alpha\left(-w(\log t)^\alpha\right) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) H\left(s,\overline{x}(s)\right)\frac{ds}{s}.\end{aligned}$$

It is clear that $\overline{x}$ is the upper solution of (1.1).

Now, let

$$\mathfrak{D}_1^\alpha \underline{x}(t) + w\underline{x}(t) = \lambda_1,\ \underline{x}(1) = x_0.$$

which has also a positive solution

$$\begin{aligned}\underline{x}(t) &= x_0 E_\alpha\left(-w(\log t)^\alpha\right) + \lambda_1 \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right)\frac{ds}{s} \\ &= x_0 E_\alpha\left(-w(\log t)^\alpha\right) + \lambda_1 (\log t)^\alpha E_{\alpha,\alpha+1}\left(-w(\log t)^\alpha\right).\end{aligned}$$

By (3.7) and the same way that we used to search the upper solution, we conclude also that $\underline{x}$ is the lower solution of (1.1). Therefore, from Theorem 3.2, we conclude that the problem (1.1) has at least one positive solution $x \in \mathcal{E}$ which verifies the inequalities (3.5) and (3.6). □

**Corollary 3.4.** *Assume that $f : [1,e]\times[0,\infty) \to [a,\infty)$ is continuous where $a$ is a positive constant such that*

$$a < \lim_{x\to+\infty} f(t,x) < +\infty, \tag{3.9}$$

*then the problem (1.1) has at least one positive solution $x$ in $\mathcal{E}$.*

*Proof.* By (3.9), there exist positive constants $N$ and $R$ such that

$$f(t,x) \leq N \text{ for any } x \geq R,\ t \in [1,e]. \tag{3.10}$$

Let $C = \max_{1\leq t\leq e, 0\leq x\leq R} f(t,x)$. Then, by (3.10), we have

$$a \leq f(t,x) \leq N + C \text{ for any } x \geq 0,\ t \in [1,e].$$

Thus, from Corollary 3.3, the problem (1.1) has at least one positive solution $x$ in $\mathcal{E}$ which satisfies the following inequalities

$$x(t) \geq x_0 E_\alpha\left(-w(\log t)^\alpha\right) + a(\log t)^\alpha E_{\alpha,\alpha+1}\left(-w(\log t)^\alpha\right),$$

and

$$x(t) \leq x_0 E_\alpha\left(-w(\log t)^\alpha\right) + (N+C)(\log t)^\alpha E_{\alpha,\alpha+1}\left(-w(\log t)^\alpha\right).$$

□

**Corollary 3.5.** *Assume that $f : [1,e]\times[0,\infty) \to [a,\infty)$ is continuous where $a$ is a positive constant and there exist $c,d > 0$, such that*

$$\max\left\{f(t,x) : (t,x) \in [1,e]\times[0,d]\right\} \leq c\Gamma(\alpha+1) - x_0. \tag{3.11}$$

*Then the problem (1.1) has at least one bounded positive solution $x$ in $\mathcal{E}$.*

*Proof.* By (3.11), we have

$$a \leq f(t,x) \leq c\Gamma(\alpha+1) - x_0 \quad \text{for any } (t,x) \in [1,e] \times [0,d].$$

Hence, from Corollary 3.3, we conclude directly that the problem (1.1) has at least one positive solution $x$ in $\mathcal{E}$ satisfying

$$0 \leq x \leq c.$$

□

**Proposition 3.6.** *Assume that* $f : [1,e] \times [0,\infty) \to [a,\infty)$ *is continuous where* $a$ *is a positive constant such that*

$$a < \lim_{x\to+\infty} \max_{1\leq t\leq e} \frac{f(t,x)}{x} < +\infty. \tag{3.12}$$

*Then the problem (1.1) has at least one bounded positive solution* $x$ *in* $\mathcal{E}$.

*Proof.* Since (3.12) is verified, there exist positive constants $M$ and $R$ such that

$$f(t,x) \leq Mx \quad \text{for any } x \geq R,\ t \in [1,e].$$

Let $C = \max_{0\leq t\leq 1, 0\leq x\leq R} f(t,x)$. Then,

$$a \leq f(t,x) \leq Mx + C \quad \text{for any } x \geq 0,\ t \in [1,e].$$

Then, we have

$$H(t,x) \leq Mx + C \quad \text{for any } x \geq 0,\ t \in [1,e]. \tag{3.13}$$

Consider the following equation

$$\mathfrak{D}_1^\alpha \overline{x}(t) + w\overline{x}(t) = M\overline{x} + C, \quad \overline{x}(1) = x_0, \tag{3.14}$$

which has the following positive solution

$$\begin{aligned}\overline{x}(t) &= x_0 E_\alpha\left(-w(\log t)^\alpha\right) \\ &\quad + \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log\frac{t}{s}\right)^\alpha\right)\left(M\overline{x}(s) + C\right)\frac{ds}{s}.\end{aligned}$$

Let us define the operator $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ as follows

$$\begin{aligned}\mathcal{A}\overline{x}(t) &= x_0 E_\alpha\left(-w(\log t)^\alpha\right) \\ &\quad + \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log\frac{t}{s}\right)^\alpha\right)\left(M\overline{x}(s) + C\right)\frac{ds}{s},\end{aligned} \tag{3.15}$$

which is completely continuous from Lemma 3.1.

Denote

$$\Omega_r = \left\{\overline{x} \in \mathcal{E} : \|\overline{x} - v_0\| \leq r < \infty\right\},$$

where $v_0(t) = C(\log t)^\alpha E_{\alpha,\alpha+1}\left(-w(\log t)^\alpha\right)$ and $r$ satisfies

$$r \geq \frac{x_0\Gamma(\alpha+1)^2 + MC}{\Gamma(\alpha+1)\left(\Gamma(\alpha+1) - M\right)}.$$

For any $\overline{x} \in \Omega_r$, we have

$$\|\overline{x}\| \leq r + \frac{C}{\Gamma(\alpha+1)}.$$

Then

$$\begin{aligned}
&\left|\left(\mathcal{A}\overline{x}\right)(t)-v_0(t)\right| \\
&=\left|x_0E_\alpha\left(-w\left(\log t\right)^\alpha\right)+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-w\left(\log\frac{t}{s}\right)^\alpha\right)M\overline{x}(s)\frac{ds}{s}\right| \\
&\le x_0E_\alpha\left(-w\left(\log t\right)^\alpha\right)+M\left\|\overline{x}\right\|\left(\log t\right)^\alpha E_{\alpha,\alpha+1}\left(-w\left(\log t\right)^\alpha\right) \\
&\le x_0+\frac{M\left(r+\frac{C}{\Gamma(\alpha+1)}\right)}{\Gamma(\alpha+1)}\left(\log t\right)^\alpha.
\end{aligned}$$

Thus

$$\left\|\mathcal{A}\overline{x}-v_0\right\|\le x_0+\frac{Mr}{\Gamma(\alpha+1)}+\frac{MC}{\Gamma(\alpha+1)^2}\le r.$$

According to the Schauder fixed point theorem, the operator $\mathcal{A}$ has at least one fixed point in $\Omega_r$. Hence, the problem (3.14) has at least one positive solution in $\Omega_r$, and by (3.13), we have

$$\begin{aligned}
\overline{x}(t)&\ge x_0E_\alpha\left(-w\left(\log t\right)^\alpha\right) \\
&\quad+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-w\left(\log\frac{t}{s}\right)^\alpha\right)H\left(s,\overline{x}(s)\right)\frac{ds}{s},\quad \overline{x}\in\Omega_r.
\end{aligned}$$

Obviously, $\overline{x}$ is the upper solution of (1.1), and by Corollary 3.3, we conclude also that

$$\underline{x}(t)=x_0E_\alpha\left(-w\left(\log t\right)^\alpha\right)+a\left(\log t\right)^\alpha E_{\alpha,\alpha+1}\left(-w\left(\log t\right)^\alpha\right),$$

is the lower solution of (1.1). From Theorem 3.2, the problem (1.1) has at least one positive solution $x$ in $\mathcal{E}$. □

## 4. Uniqueness of positive solutions

In this section, we shall prove the uniqueness of the positive solution using the contraction mapping principle.

**Theorem 4.1.** *Assume that* $f:[1,e]\times[0,\infty)\to[0,\infty)$ *is continuous and there exists* $L>0$ *such that*

$$\left|f(t,x_1)-f(t,x_2)\right|\le L\left|x_1-x_2\right| \text{ for any } t\in[1,e],\ x_1,x_2\in[0,\infty), \tag{4.1}$$

*with*

$$\frac{L}{\Gamma(\alpha+1)}<1. \tag{4.2}$$

*Then the problem (1.1) has a unique positive solution* $x\in\mathcal{K}$.

*Proof.* From Theorem 3.2, it follows that (1.1) has at least one positive solution in $\mathcal{K}$. Hence, we need only to prove that the operator $\mathcal{A}$ defined in (3.1) is a contraction on $\mathcal{K}$. In fact, since for any $x_1,x_2\in\mathcal{K}$, (4.1) and (4.2) are verified, then we have

$$\begin{aligned}
&\left|\left(\mathcal{A}x_1\right)(t)-\left(\mathcal{A}x_2\right)(t)\right| \\
&\le\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-w\left(\log\frac{t}{s}\right)^\alpha\right)\left|f\left(s,x_1(s)\right)-f\left(s,x_2(s)\right)\right|\frac{ds}{s} \\
&\le\frac{\left(\log t\right)^\alpha}{\Gamma(\alpha+1)}L\left\|x_1-x_2\right\| \\
&\le\frac{L}{\Gamma(\alpha+1)}\left\|x_1-x_2\right\|.
\end{aligned}$$

Thus,

$$\|\mathcal{A}x_1 - \mathcal{A}x_2\| \leq \frac{L}{\Gamma(\alpha+1)} \|x_1 - x_2\|.$$

Hence, the operator $\mathcal{A}$ is a contraction mapping by (4.2). Therefore, by the contraction mapping principle, we conclude that the problem (1.1) has a unique positive solution $x \in \mathcal{K}$. □

Finally, we give an example to illustrate our results.

**Example 4.1.** We consider the following nonlinear fractional relaxation differential equation

$$\begin{cases} \mathfrak{D}_1^{\frac{1}{5}} x(t) + x(t) = \dfrac{1}{2+t}\left(\dfrac{t-1}{x(t)+1} + 2\right), \ 1 < t \leq e, \\ x(1) = 1, \end{cases} \tag{4.3}$$

where $\alpha = 1/5$, $w = 1$, $x_0 = 1$ and $f(t,x) = \dfrac{1}{2+t}\left(\dfrac{t-1}{x+1} + 2\right)$. Since $f$ is continuous and

$$\frac{2}{2+e} \leq f(t,x) \leq \frac{2}{3},$$

for $(t,x) \in [1,e] \times [0,\infty)$, hence by Corollary 3.3, (4.3) has a positive solution which verifies $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ where

$$\overline{x}(t) = E_{1/5}\left(-(\log t)^{1/5}\right) + \frac{2}{3}(\log t)^{1/5} E_{1/5,6/5}\left(-(\log t)^{1/5}\right),$$

and

$$\underline{x}(t) = E_{1/5}\left(-(\log t)^{1/5}\right) + \frac{2}{2+e}(\log t)^{1/5} E_{1/5,6/5}\left(-(\log t)^{1/5}\right),$$

are respectively the upper and lower solutions of (4.3). Also, we have

$$\frac{L}{\Gamma(\alpha+1)} \simeq 0.397 < 1,$$

then by Theorem 4.1, (4.3) has a unique positive solution which is bounded by $\underline{x}$ and $\overline{x}$.

## 5. Conclusion

In the current paper, we have studied the existence and uniqueness of positive solutions for nonlinear Caputo-Hadamard fractional relaxation differential equations. We have presented the existence and uniqueness theorems for the problem (1.1) under some sufficient conditions due to the Schauder and Banach fixed point theorems and the method of upper and lower solutions. The main results have been well illustrated with the help of an example. Our results in this paper have been extended some wellknown results. It seems that the results of this paper can be extended to cover the case of delay fractional relaxation differential equations.

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