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PRIME AND IRREDUCIBLE UP-FILTERS OF MEET-COMMUTATIVE UP-ALGEBRAS

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ABSTRACT. In this article, we give a number of important properties of meet-commutative UP-algebras. In the class of these UP-algebras, we introduce and analyze the concepts of prime and irreducible UP-filters.

1. INTRODUCTION

The concept of KU-algebras was introduced in 2009 by Prabpayak and Leerawat in the article [5]. Iampan introduced the concept of UP-algebras as a generalization of KU-algebras ([1]). In [7], Somjanta et al. introduced the notion of UP-filters in this class of algebra. Jun and Iampan then introduced and analyzed several classes of filters in UP-algebras such as implicative, comparative and shift UP-filters (see [2, 3, 4]). In [3] it is shown that comparative UP-filter and implicative UP-filter do not have to be coincide. Also, in [3] it is proved (Theorem 3) that if a meet-commutative UP-algebra, i.e. UP-algebra A that satisfies the condition

$$(\forall x, y \in A)((x \cdot y) \cdot y = (y \cdot x) \cdot x),$$

satisfies an additional condition

$$(\forall x, y, z \in A)(x \cdot (y \cdot z) = y \cdot (x \cdot z)),$$

then the comparative UP filters and the implicative UP filters are coincide.

The concept of meet-commutative UP-algebras was introduced in article [6] in a different context than in paper [3]. This seems to justify our interest in studying the properties of UP-algebras in which the property of meet-commutativity is present. In this article, we give a number of important properties of meet-commutative UP-algebras. In addition, in the class of meet-commutative UP-algebras, we introduce and analyze the concepts of prime and irreducible UP-filters.

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Goals of the manuscript are:

- More attention is paid to determining the fundamental properties of meet-commutative UP-algebras;
- Determine the concepts of prime and irreducible UP-filters in such UP-algebras;
- Finding the conditions of existence of and the connection between of these two classes of UP-filters.

The novelty in the manuscript (according to our sincere belief) is the complete material presented in section 3. In subsection 3.1, we especially emphasize Theorem 3.5 in which the existence of the least upper bound in a meet-commutative UP-algebra is proved. In subsection 3.2, the definitions of the notion of 'prime UP-filters' and the concept of 'irreducible UP-filters' in a meet-commutative UP-algebra appears for the first time (Definition 3.8 and Definition 3.16). Of course, claims and examples about the properties of prime UP-filters and the relationship between prime UP-filters and irreducible UP-filters appear for the first time, also.

2. Preliminaries

In this section, taking from the literature, we will repeat some concepts and statements of interest for this research. The way of writing formulas in this text is a strictly logical way of writing. So we tried to get the formulas written in the standard way of writing formulas in logic. For example, the signs '$\wedge$' and '$\vee$' are logical conjunction and disjunction. In addition, we use the symbol ':=' to indicate abbreviations in a sense such as $S := \{u \in A : P(u)\}$ where the symbol S denotes the set determined by predicate P on the right side of the notation ':='.

An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* (see [1]) if it satisfies the following axioms:

(UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,

(UP-2) $(\forall x \in A)(0 \cdot x = x)$,

(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$,

(UP-4) $(\forall x, y \in A)((x \cdot y = 0 \,\wedge\, y \cdot x = 0) \implies x = y)$.

Example 2.1. ([1], Example 1.6) Let $A = \{0, a, b, c\}$ and operation '$\cdot$' is defined on A as follows:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Then $A = (A, \cdot, 0)$ is a UP-algebra.

A UP-algebra A is said to be meet-commutative (see [3], Definition 3) if it satisfies the condition

(20) $(\forall x, y \in A)((x \cdot y) \cdot y = (y \cdot x) \cdot x)$.

However, this term also appears (Definition 1.15) in the paper [6].

In a UP-algebra, the order relation '$\leqslant$' is defined as follows

$$(\forall x, y \in A)(x \leqslant y \iff x \cdot y = 0).$$

A subset F of a UP-algebra A is called a *UP-filter* of A (see [7]) if it satisfies the following conditions:

(F-1) $0 \in F$,
(F-2) $(\forall x, y \in A)((x \in F \wedge x \cdot y \in F) \implies y \in F)$.

It is clear that every UP-filter F of a UP-algebra A satisfies:
(1) $(\forall x, y \in A)((x \in F \wedge x \leqslant y) \implies y \in F)$.

The family of all UP-filters of a UP-algebra A is denoted by $\mathfrak{F}(A)$. It is easy to verify that the intersection of UP-filters of a UP-algebra A is also a UP-filter of A. For any subset S of A, let $F(S) := \bigcap\{F \in \mathfrak{F}(A) : S \subseteq F\}$. Then $F(S)$ is the smallest UP-filter of A containing S.

Definition 2.2. ([2], Definition 1) A subset F of a UP-algebra A is called an *implicative UP-filter* of A if it satisfies the following conditions:
(F-1) $0 \in F$ and
(IF) $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in F \wedge x \cdot y \in F) \implies x \cdot z \in F)$.

Definition 2.3. ([3], Definition 2) A subset F of a UP-algebra A is called a *comparative UP-filter* of A if it satisfies the following conditions:
(F-1) $0 \in F$ and
(CF) $(\forall x, y, z \in A)((x \cdot ((y \cdot z) \cdot y) \in F \wedge x \in F) \implies y \in F)$.

In the general case, a comparative UP-filter does not have to be an implicative UP-filter and vice versa.

Example 2.4. Let A be as in Example 2.1. By direct versification it is routine to verify that subsets $\{0\}$, $\{0; b\}$, $\{0, c\}$ and $\{0, b, c\}$ are implicative UP-filters of A (see [2]). On the other hand, $F := \{0, b\}$ is not a comparative UP-filter of A. since $b \cdot ((c \cdot a) \cdot c) = b \cdot (a \cdot c) = b \cdot 0 = 0 \in F$ and $b \in F$ but $c \notin F$.

Example 2.5. ([3], Example 1) Let $A = \{0, a, b, c\}$ and operation '$\cdot$' is defined on A as follows:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	a	b
b	0	0	0	b
c	0	a	0	0

Then $A = (A, \cdot, 0)$ is a UP-algebra and the subset $F := \{0, a, b\}$ are comparative UP-filter of A but it is not an implicative UP-filter of A since for example for $x = 0$, $y = a$ and $z = c$, we have $x \cdot (y \cdot z) = 0 \cdot (a \cdot c) = 0 \cdot b = b \in F$ and $x \cdot y = 0 \cdot a = a \in F$ but $x \cdot z = 0 \cdot c = c \notin F$.

In [3], the conditions for a comparative UP-filter to be an implicative UP-filter are described.

Theorem 2.1 ([3], Theorem 3)**.** *If a meet-commutative UP-algebra A satisfies the condition*
(21) $(\forall x, y, z \in A)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$,
then comparative UP-filters and implicative UP-filters coincide.

In what follows, the following result is important.

Theorem 2.2 ([2], Theorem 6)**.** *Let A be a UP-algebra satisfying the condition (21). If F is an implicative UP-filter of A, then $(F \cup \{a\}] = \{x \in A : a \cdot x \in F\}$ for all $a \in A$.*

3. Prime UP-filters and some properties of meet-commutative UP-algebras

This section contains the main part of this research. After expressing some of the basic properties of meet-commutative UP-algebras, we introduce and analyze the concepts of prime and irreducible UP-filters.

3.1. **Some properties of meet-commutative UP-algebras.** We begin this subsection with the following example.

Example 3.1. Let $A = \{0, a, b\}$ and operation '$\cdot$' is defined on A as follows:

$\cdot$	0	a	b
0	0	a	b
a	0	0	a
b	0	0	0

Then $A = (A, \cdot, 0)$ is a meet-commutative UP-algebra ([6], Example 1.16). By direct verification it can be verified that this meet-commutative UP-algebra satisfies the condition (21).

The following example shows that any UP-algebra may not satisfy the condition (20).

Example 3.2. ([3], Example 3) Let $A = \{0, a, b, c\}$ and operation '$\cdot$' is defined on A as follows:

$\cdot$	1	a	b	c
0	0	a	b	c
a	0	0	a	a
b	0	0	0	a
c	0	0	0	0

Then $A = (A, \cdot, 0)$ is a UP-algebra which does not satisfy the condition (20). For example, for $x = b$ and $y = c$, we have $(x \cdot y) \cdot y = (b \cdot c) \cdot c = a \cdot c = a$ but $(y \cdot x) \cdot x = (c \cdot b) \cdot b = 0 \cdot b = b$.

The following example shows that, in the general case, a UP-algebra does not satisfies the condition (21).

Example 3.3. Let $A = \{0, a, b, c, d\}$ and operation '$\cdot$' is defined on A as follows:

$\cdot$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	0	0	0
b	0	b	0	0	0
c	0	b	b	0	0
d	0	b	b	c	0

Then $A = (A, \cdot, 0)$ is a UP-algebra ([1], Example 1.12). Since $c \cdot (b \cdot a) = b \cdot b = b$ but $b \cdot (c \cdot a) = b \cdot b = 0$ we have that this UP-algebra does not satisfy the condition (21).

We first characterize the meet-commutative UP-algebras.

Theorem 3.1. *Let A be a meet-commutative UP-algebra. Then the following holds*

(22) $(\forall x, y \in A)(x \leqslant y \implies y = (y \cdot x) \cdot x)$.

Proof. Let $x, y \in A$ be such that $x \leqslant y$. Then $x \cdot y = 0$. Thus $y = 0 \cdot y = (x \cdot y) \cdot y = (y \cdot x) \cdot x$. □

Theorem 3.2. *Let A be a meet-commutative UP-algebra. For any $x, y \in A$, the element $x \sqcup y := (x \cdot y) \cdot y = (y \cdot x) \cdot x$ is the least upper bound of x and y.*

Proof. It is clear that the element $x \sqcup y$ is a common upper bound of x and y because the following

$$x \leqslant (y \cdot x) \cdot x, \ y \leqslant (x \cdot y) \cdot y \ \text{ and } (x \cdot y) \cdot y = (y \cdot x) \cdot x$$

holds. Let z be a common upper bound for x and y, i.e. let $x \leqslant z$ and $y \leqslant z$ be valid. Then $z = (z \cdot x) \cdot x$ and $z = (z \cdot y) \cdot y$ by (22). Thus

$$z = (z \cdot x) \cdot x = (((z \cdot y) \cdot y) \cdot x) \cdot x.$$

On the other hand, from $y \leqslant (z \cdot y) \cdot y$ it follows $((z \cdot y) \cdot y) \cdot x \leqslant y \cdot x$ according to the claim (5) of Propositions 1.8 in [1]. Applying this procedure again, we obtain

$$(y \cdot x) \cdot x \leqslant (((z \cdot y) \cdot y) \cdot x) \cdot x = z.$$

Therefore, $x \sqcup y$ is the least upper bound of x and y. □

Corollary 3.3. *If A is a meet-commutative UP-algebra, then $(A, \sqcup)$ is the upper semilattice.*

Proposition 3.4. *Let A be a meet-commutative UP-algebra. Then*

(23) $(\forall x, y \in A)(0 \sqcup x = x, \ x \sqcup 0 = 0, \ x \sqcup x = x, \ \textit{and } x \sqcup y = y \sqcup x)$.

(24) $(\forall x, y, z \in A)((x \sqcup y) \sqcup z \leqslant (x \sqcup z) \sqcup (y \sqcup z))$.

(25) $(\forall x, y, z \in A)((z \cdot x) \sqcup (z \cdot y) \ \leqslant \ z \cdot (x \sqcup y))$.

(26) $(\forall x, y, z \in A)((x \sqcup y) \cdot z \ \leqslant \ (x \cdot z) \sqcup (y \cdot z))$.

(27) $(\forall x, y \in A)(x \sqcup y \leqslant (y \cdot x) \sqcup (x \cdot y))$.

Proof. (23) Apparently by definition of the least upper bound and with respect to the condition (UP-3). (For proof of these equations, a reader may also refer to Proposition 2.3. in [6].)

(24) From $x \leqslant x \sqcup z$ and $y \leqslant y \sqcup z$ immediately it follows $x \sqcup y \leqslant (x \sqcup z) \sqcup (y \sqcup z)$ according to the definition of the term least upper bound. On the other hand, we have that $z \leqslant (x \sqcup z) \sqcup (y \sqcup z)$ also is valid. This, together with the previous inequality, it gives the inequality (24).

(25) For the given elements x and y, we have $x \leqslant x \sqcup y$ and $y \leqslant x \sqcup y$ according to the claim (6) of Proposition 1.8 in [1]. Then $z \cdot x \leqslant z \cdot (x \sqcup y)$ and $z \cdot y \leqslant z \cdot (x \sqcup y)$ by (4) of Proposition 1.8 in [1]. Thus $(z \cdot x) \sqcup (z \cdot y) \leqslant z \cdot (x \sqcup y)$.

(26) From the valid inequalities $x \leqslant x \sqcup y$ and $y \leqslant x \sqcup y$, according to (5) of Proposition 1.8 in [1], we have $(x \sqcup y) \cdot z \leqslant x \cdot z$ and $(x \sqcup y) \cdot z \leqslant y \cdot z$. Hence $(x \sqcup y) \cdot z \ \leqslant \ (x \cdot z) \sqcup (y \cdot z)$.

(27) From the valid inequalities $x \leqslant y \cdot z$ and $y \leqslant x \cdot y$ we directly obtain the required inequality according to the definition of the concept of least upper bound. □

3.2. **Concept of prime UP-filters.** The following definition introduces the concept of prime UP-filters of a meet-commutative UP-algebra.

Definition 3.4. Let F be a UP-filter of a meet-commutative UP-algebra A. Then F is said to be a *prime UP-filter* of A if the following holds

(PF) $(\forall x, y \in A)(x \sqcup y \in F \implies (x \in F \ \vee \ y \in F))$.

Example 3.5. Let $A = \{0, a, b, c\}$ and operation '$\cdot$' is defined on A as follows:

·	1	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	c	0	c
c	0	b	b	0

Then $A = (A, \cdot, 0)$ is a meet-commutative UP-algebra. Subsets $\{0\}$, $\{0, b\}$ and $\{0, c\}$ are UP-filters of A. It is not difficult to verify that UP filters $\{0, b\}$ and $\{0, c\}$ are prime. It is clear that $\{0\}$ is not a prime IP-filter of A because $b \sqcup c = 0 \in \{0\}$ but $b \notin \{0\}$ and $c \notin \{0\}$.

It follows directly from the previous definition.

Proposition 3.5. *Let F be a prime UP-filter of a meet-commutative UP-algebra A. Then*

$$(\forall x, y \in A)(x \sqcup y \in F \implies (x \cdot y \in F \vee y \cdot x \in F)).$$

Proof. Proof of this proposition it follows directly from the claim (27). □

A characteristic of prime UP-filters of a meet-commutative UP-algebra is given in the following theorems.

Theorem 3.6. *Let F be a UP-filter of a meet-commutative UP-algebra A. If the following holds*

(E) $(\forall x, y \in A)(x \cdot y \in F \vee y \cdot x \in F)$,

then F is a prime UP-filter of A.

Proof. Let a UP-filter F satisfy the condition (E) and let $x, y \in A$ be such that $x \sqcup y \in F$ are valid. Then $(x \cdot y) \cdot y = (y \cdot x) \cdot x \in F$. On the other hand, according to (E), we have $x \cdot y \in F$ or $y \cdot x \in F$. If $x \cdot y \in F$, then from $x \cdot y \in F$ and $(x \cdot y) \cdot y \in F$ it follows $y \in F$ by (F-2). If $y \cdot x \in F$, then from $y \cdot x \in F$ and $(y \cdot x) \cdot x \in F$ it follows $x \in F$ by (F-2). Hence (PF) holds. □

Corollary 3.7. *Let F and G be UP-filters of a meet-commutative UP-algebra A such that $F \subseteq G$. If the UP filter F satisfies the condition (E), then G is a prime UP- filter.*

Proof. Since F is a prime UP-filter of A that satisfies the condition (E), it follows that the UP-filter G also satisfies the condition (E). Therefore G is a prime UP-filter of A according to the previous theorem. □

Corollary 3.8. *If the order relation in a meet-commutative UP-algebra A is a linear relation, then each UP-filter of A is a prime UP-filter of A.*

Proof. Suppose that the order relation in a meet-commutative UP-algebra A is a linear order. Then for each elements $x, y \in A$ the following $x \leqslant y$ or $y \leqslant x$ holds. Thus $x \cdot y = 0$ or $y \cdot x = 0$. Hence, for every UP-filter F of A, we have either $x \cdot y \in F$ or $y \cdot x \in F$ by (F-1). This shows that F is a prime UP-filter of A in accordance to Theorem 3.6. □

Theorem 3.9. *Let F be a prime UP-filter of a meet-commutative UP-algebra A. Then:*

(i) $(\forall x, y \in A)(x \sqcup y = 0 \implies (x \in F \vee y \in F))$.

(ii) $(\forall x, y, z \in A)((x \sqcup y) \sqcup z \in F \implies (x \sqcup z \in F \vee y \sqcup z \in F))$.

Proof. Assume that F is a prime UP-filter of A.

Let x, y be arbitrary elements of A such that $x \sqcup y = 0$. Then $x \sqcup y \in F$ by (F-1). Thus $x \in F$ or $y \in F$ by (PF). Hence (i) is valid.

The claim (ii) is a direct consequence from (24) with respect to (1). □

The question it is expected to be asked is: Does a prime UP-filter of a meet-commutative UP-algebra satisfy condition (E)? The following proposition shows one of the possible answers to this question.

Proposition 3.10. *If a meet-commutative UP-algebra A satisfies*
(U) $(\forall x, y \in A)((x \cdot y) \sqcup (y \cdot x) = 0)$,
then any prime UP-filter of A satisfies condition (E).

Proof. The proof of this proposition follows directly from the definition of prime UP-filters and the statement (i) of the previous theorem. □

It is obvious that if the order relation in a meet-commutative UP-algebra A is a linear order, then A satisfies the condition (U).

3.3. **Concept of irreducible UP-filters.**

Definition 3.6. A UP-filter F of a UP-algebra A is said to be an *irreducible UP-filter* of A if for any UP-filters S and T of A the following implication holds

$$F = S \cap T \implies (S = F \vee T = F).$$

Theorem 3.11. *Any prime UP-filter of a meet-commutative UP-algebra is an irreducible UP-filter.*

Proof. Suppose F is a prime UP-filter of a UP-algebra A. Let S and T be UP-filters of A such that $F = S \cap S$. Let us assume that the elements $a, b \in A$ be such that $a \in S$ and $b \in T$ holds. Since $a \leqslant a \sqcup b$ and $b \leqslant a \sqcup b$ holds, it follows $a \sqcup b \in S$ and $a \sqcup b \in T$ because S and T are UP-filters of UP-algebra A. Then $a \sqcup b \in S \cap R = F$. From here, it follows $a \in F$ or $b \in F$ because F is a prime UP-filter of A. Since this inclusion holds for arbitrary elements $a \in S$ and $b \in T$, we conclude that $S \subseteq F$ or $T \subseteq F$ holds. As the inverse inclusion certainly holds, we conclude that $S = F$ or $T = F$ holds. □

Remark. *In fact, it is enough to prove the claim 'If for UP-filters S, T and F of a meet-commutative UP-algebra A such that $S \cap T \subseteq F$ and F is prime, then $S \subseteq F$ or $T \subseteq F$ holds' since from this implication it follows that F is an irreducible UP-filter of A.*

Example 3.7. Let A be as in Example 3.5. It is obvious that the equality $\{0\} = \{0, b\} \cap \{0, c\}$ is valid since $\{0\}$ is not a prime UP-filter of A.

It is quite natural to ask when an irreducible UP-filter of a meet-commutative UP-algebra A is a prime UP-filter. One of possible answers to this question is offered by the following analysis.

Let a meet-commutative UP-algebra A satisfy the condition (21), F be an irreducible implicative UP-filter of A and let $x, y \in A$ be such $x \sqcup y \in F$. Then $(y \cdot x) \cdot x = (x \cdot y) \cdot y \in F$. This means

$$x \in \{u \in A : (y \cdot x) \cdot u \in F\} \text{ and } y \in \{v \in A : (x \cdot y) \cdot v \in F\}.$$

On the other hand, obviously the following formula is valid formula

$$F \subseteq (F \cup \{x \cdot y\}] \cap (F \cup \{y \cdot x\}] = \{u \in A : (x \cdot y) \cdot u \in F\} \cap \{v \in A : (y \cdot x) \cdot v \in F\}$$

according to Theorem 2.2. If the following

$$\{u \in A : (x \cdot y) \cdot u \in F\} \cap \{v \in A : (y \cdot x) \cdot v \in F\} \subseteq F$$

could be valid, then we would have

$$y \in \{u \in A : (x \cdot y) \cdot u \in F\} = F \text{ or } x \in \{v \in A : (y \cdot x) \cdot v \in F\} = F.$$

This would mean that the UP-filter F satisfies the condition (PE) and would therefore be a prime UP-filter of A. the bodytext. This is the bodytext. This is the bodytext. This is the bodytext. This is the bodytext.

4. Conclusions

The concept of meet-commutative UP-algebras was introduced in the article [6] by Sawika et al. In such UP-algebras, the comparative and the implicative UP-filters coincide. In this paper, we deal in more detail with internal structures of meet-commutative UP-algebras. It is shown that the least upper bound '$\sqcup$' for any two elements can be designed in it. In subsections 3.2 and 3.3 two new concepts have been introduced in UP-algebras: the concept of prime UP-filters and the concept of irreducible UP-filters and some of their important properties have been established.

In future work, it is possible, among other things, to consider the existence of some other types of prime UP-filters in meet-commutative UP-algebras as well as their interconnections. Particular attention can be paid to finding conditions that would allow to coincide of prime UP-filters of several kinds in such UP-algebras.

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