ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 4, Number 1 (2021), 10-16
ISSN: 2582-0818
© http://www.technoskypub.com

WEAKLY IRREDUCIBLE UP-FILTERS IN MEET-COMMUTATIVE UP-ALGEBRAS

DANIEL A. ROMANO

ABSTRACT. The concept of meet-commutative UP-algebras was introduced in 2016 by Sawika et al. Muhiuddin et al. introduced in 2021 the concept of prime UP-filter (of the first kind) and irreducible UP-filter in meet-commutative UP-algebras. Also, it has been shown that any prime UP-filter in such algebras is irreducible. In this paper, we introduce the concept of weakly irreducible UP-filters in such algebras and show that the prime UP-filter is between this and the irreducible UP-filter. Also, we show the possibility that each irreducible UP-filter is a weakly irreducible UP-filter.

1. INTRODUCTION

The concept of KU-algebras was introduced in 2009 by C. Prabpayak and U. Leerawat in the article [9]. In 2017, A Iampan u [3] introduced the concept of UP-algebras as a generalization of KU-algebras. In [17], Somjanta et al. introduced the notion of filters in this class of algebras. Jun and Iampan then introduced and analyzed several classes of filters in UP algebras such as implicative, comparative and shift UP-filters (see, for example, [4, 5, 6]). Romano also took part in the analysis of filter types in such algebras as proper UP-filter [11] and (with Y. B. Jun) weak implicative UP-filter [14].

The concept of meet-commutative UP-algebras was introduced in article [15] by Sawika et al. In such algebras, Muhiuddin et al. introduced the concepts of prime (of the first kind) and irreducible UP-filters [8]. In that paper it is also shown that any prime UP-filter in meet-commutative UP-algebra is an irreducible UP-filter. Continuing to develop the ideas introduced in the mentioned paper [8], Romano introduced the concepts of prime UP-filters of the second [12] and the third kind [13]. This seems to justify the author's interest in studying the properties of UP-algebras in which the property of meet-commutativity is present.

2010 *Mathematics Subject Classification.* 03G25, 06F35.

Key words and phrases. UP-algebra, meet-commutative UP-algebra, prime UP-filter, irreducible UP-filter, weak irreducible UP-filters.

Received: January 1, 2021. Accepted: March 7, 2021. Published: March 31, 2021.

A UP-filter F of a meet-commutative UP-algebra A is said to be an *irreducible UP-filter* of A if for any UP-filters S and T of A the following implication holds

$$F = S \cap T \implies (S = F \vee T = F).$$

In an effort to better understand the properties of this class of logical algebras, the author analyzed the possibility of weakening the hypothesis in the previous implication. The consequence of the $S \cap T \subseteq F$ option is considered. As a consequence of this choose, a new class of UP-filters in meet-commutative UP-algebras was obtained.

In this paper, the author introduces the concept of weakly irreducible UP-filters in meet-commutative UP-algebras (Definition 4.1), gave one important characterization of such UP-filter (Theorem 4.1) and connects it with irreducible (Theorem 4.3 and Theorem 4.4) and prime UP-filters (Theorem 4.2).

2. Preliminaries

This section introduces the concepts and processes with them that will be used in the main section.

2.1. UP-algebras.

In this subsection, taking from the literature, we will repeat the logical environment of interest for this research.

An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* (see [3]) if it satisfies the following axioms:

(UP-1) $(\forall x, y, \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2) $(\forall x \in A)(0 \cdot x = x)$,
(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$,
(UP-4) $(\forall x, y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \implies x = y)$.

Example 2.1. ([3], Example 1.6) Let $A = \{0, a, b, c\}$ and operations '$\cdot$' defined on A as follows:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Then $A = (A, \cdot, 0)$ is a UP-algebra where.

In a UP-algebra, the order relation '$\leqslant$' is defined as follows

$$(\forall x, y \in A)(x \leqslant y \iff x \cdot y = 0).$$

2.2. UP-filters.

A subset F of a UP-algebra A is called a *UP-filter* of A (see [17]) if it satisfies the following conditions:

(F-1) $0 \in F$,
(F-2) $(\forall x, y \in A)((x \in F \wedge x \cdot y \in F) \implies y \in F)$.

It is clear that every UP-filter F of a UP-algebra A satisfies:
(1) $(\forall x, y \in A)((x \in F \wedge x \leqslant y) \implies y \in F)$.

The family of all UP-filters of a UP-algebra A is denoted by $\mathfrak{F}(A)$. It is easy to verify that the intersection of UP-filters of a UP-algebra A is also a UP-filter of A. For any subset S of A, let $F(S) := \bigcap\{F \in \mathfrak{F}(A) : S \subseteq F\}$. Then $F(S)$ is the smallest UP-filter of A containing S. Therefore, if S and T are UP-filters in a UP-algebra A, then $S \sqcap T := S \cap T$ and $S \sqcup T := F(S \cup T)$ are UP-filters in A. So, $(\mathfrak{F}(A), \sqcap, \sqcup)$ is a complete lattice.

2.3. **Meet-commutative UP-algebras.** A UP-algebra A is said to be meet-commutative (see [5], Definition 3) if it satisfies the condition

(2) $(\forall x, y \in A)((x \cdot y) \cdot y = (y \cdot x) \cdot x)$.

However, this term also appears (Definition 1.15) in the paper [15]. We begin this subsection with the following example.

Example 2.2. Let $A = \{0, a, b\}$ and operations '$\cdot$' defined on A as follows:

$\cdot$	0	a	b
0	0	a	b
a	0	0	a
b	0	0	0

Then $A = (A, \cdot, 0)$ is a meet-commutative UP-algebra ([15], Example 1,16).

Example 2.3. ([16], Example 2.2) Let $A = \{0, a, b, c, d, e\}$ and operations '$\cdot$' defined on A as follows:

$\cdot$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	0	0	b	c	d	e
b	0	a	0	c	d	e
c	0	a	b	0	d	e
d	0	a	b	c	0	e
e	0	a	b	c	d	0

Then $A = (A, \cdot, 0)$ is a meet-commutative UP-algebra. For example, $(a \cdot b) \cdot b = (b \cdot a) \cdot a = 0$ and $(b \cdot e) \cdot e = (e \cdot b) \cdot b = 0$.

The following example shows that any UP-algebra may not satisfy the condition (2).

Example 2.4. ([5], Example 3) Let $A = \{0, a, b, c\}$ and operations '$\cdot$' defined on A as follows:

$\cdot$	1	a	b	c
0	0	a	b	c
a	0	0	a	a
b	0	0	0	a
c	0	0	0	0

Then $A = (A, \cdot, 0)$ is a UP-algebra which does not satisfy the condition (2). For example, for $x = b$ and $y = c$, we have $(x \cdot y) \cdot y = (b \cdot c) \cdot c = a \cdot c = a$ but $(y \cdot x) \cdot x = (c \cdot b) \cdot b = 0 \cdot b = b$.

We first characterize the meet-commutative UP-algebras.

Theorem 2.1 ([8])**.** *Let A be a meet-commutative UP-algebra. Then the following holds*

$$(\forall x, y \in A)(x \leqslant y \implies y = (y \cdot x) \cdot x).$$

The following theorem is crucial for understanding this logic environment and recognizing the properties of various types of UP-filters in such an environment.

Theorem 2.2 ([8])**.** *Let A be a meet-commutative UP-algebra. For any $x, y \in A$, the element*

$$x \sqcup y := (x \cdot y) \cdot y = (y \cdot x) \cdot x$$

is the least upper bound of x and y.

Theorem 2.3 ([12, 13]). *Let A be a meet-commutative UP-algebra. Then*
$(\forall x, y \in A)(0 \sqcup x = x,\ x \sqcup 0 = 0,\ x \sqcup x = x,$ *and* $x \sqcup y = y \sqcup x)$.
$(\forall x, y, z \in A)((x \sqcup y) \sqcup z = (x \sqcup z) \sqcup (y \sqcup z))$.
$(\forall x, y, z \in A)((z \cdot x) \sqcup (z \cdot y) \leqslant z \cdot (x \sqcup y))$.
$(\forall x, y, z \in A)((x \sqcup y) \cdot z \leqslant (x \cdot z) \sqcup (y \cdot z))$.
$(\forall x, y \in A)(x \sqcup y \leqslant (y \cdot x) \sqcup (x \cdot y))$.

Corollary 2.4. *If A is a meet-commutative UP-algebra, then $(A, \sqcup)$ is the upper semilattice.*

Relying on one of the results of Theorem 2.3, we conclude that $(A, \sqcup)$ is a semilattice with respect to the operation '$\sqcup$' with special distributivity property different than that the concept of 'distributive semilattice' described in the book [2], for example.

3. Prime and irreducible UP-filters

In this and the next section, we will assume that the algebra A is a meet-commutative UP-algebra. Although the material presented in this section can be found in [8], it is included in this report due to the consistency of the entire content.

Definition 3.1. ([8]) Let F a UP-filter of A. Then F is said to be a *prime UP-filter* of A if the following holds
(PF) $(\forall x, y \in A)(x \sqcup y \in F \implies (x \in F \vee y \in F))$.

Example 3.2. Let $A = \{0, a, b, c\}$ and operations '$\cdot$' defined on A as follows:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	c	0	c
c	0	b	b	0

Then $A = (A, \cdot, 0)$ is a meet-commutative UP-algebra. Subsets $\{0\}$, $\{0, b\}$, $\{0, c\}$ and $\{0, a, b, c\}$ are UP-filters of A. It is not difficult to verify that UP filters $\{0, b\}$ and $\{0, c\}$ are prime. It is clear that $\{0\}$ is not a prime IP-filter of A because $b \sqcup c = 0 \in \{0\}$ but $b \notin \{0\}$ and $c \notin \{0\}$.

Definition 3.3. ([8]) A UP-filter F of a A is said to be an *irreducible UP-filter* of A if for any UP-filters S and T of A the following implication holds
$$F = S \cap T \implies (S = F \vee T = F).$$

Theorem 3.1 ([8]). *Any prime UP-filter in a meet-commutative UP-algebra is an irreducible UP-filter.*

Proof. Suppose F is a prime UP-filter of a UP-algebra A. Let S and T be UP-filters of A such that $F = S \cap S$. Let us assume that the elements $a, b \in A$ be such that $a \in S$ and $b \in T$ holds. Since $a \leqslant a \sqcup b$ and $b \leqslant a \sqcup b$ holds, it follows $a \sqcup b \in S$ and $a \sqcup b \in T$ because S and T are UP-filters in UP-algebra A. Then $a \sqcup b \in S \cap R = F$. From here, it follows $a \in F$ or $b \in F$ because F is a prime UP-filter of A. Since this inclusion holds for arbitrary elements $a \in S$ and $b \in T$, we conclude that $S \subseteq F$ or $T \subseteq F$ holds. As the inverse inclusion certainly holds, we conclude that $S = F$ or $T = F$ holds. □

Example 3.4. Let A be as in Example 2.3. The subsets $\{0\}$, $\{0, a\}$ and A are UP-filters in A. As shown in Example 2.3, it is clear that $\{0\}$ is not a prime UP-filter in A because for elements $a, b \in A$ holds $a \sqcup b \in \{0\}$ but $a \notin \{0\}$ and $b \notin \{0\}$. The only prime UP-filter in A is just A.

4. Concept of weakly irreducible UP-filters

This and the next section form a major part of this report. First, the notion of weakkly irreducible UP-filter is introduced in a meet-commutative UP-algebra, and then one of its important characterizations is given and it is connected with irreducible and prime UP-filters.

Definition 4.1. Let F be a UP-filter of A. F is a *weakly irreducible UP-filter* in A if and only if for all UP-filter S and T of A such that $S \cap T \subseteq F$ the following holds $S \subseteq F$ or $T \subseteq F$.

Example 4.2. Let A be as in the example 3.2. Then A is a meet-commutative UP-algebra. For the UP filter $F := \{0, b\}$ we have $\{0, b\} \cap \{0, a, b, c\} \subseteq F$ and $\{0, b\} \subseteq F$. Thus, F is a weakly irreducible UP-filter in A.

The following theorem gives one important characterization of weakly irreducible UP-filters in meet-commutative UP-algebras.

Theorem 4.1. *Let F be a UP-filter in A. Then F is weakly irreducible if and only if the following conditions hold*

(WIrF) $(\forall x, y \in A)((x \notin F \wedge y \notin F) \implies (\exists z \notin F)(x \leqslant z \wedge y \leqslant z))$.

Proof. Assume that F is a weakly irreducible UP-filter in A and let elements $x, y \in A$ be such that $x \notin F$ and $y \notin F$. Let us design UP-filters $F_x = F(F \cup \{x\})$ and $F_y = F(F \cup \{y\})$. It is obvious that $F \subseteq F_x \cap F_y$. If it were $F = F_x \cap F_y$, we would have $x \in F_x \subseteq F$ or $y \in F_y \subseteq F$ which is impossible according to the hypothesis. So it has to be $F \subset F_x \cap F_y$. Thus, there exists an element $z \in F_x \cap F_y$ such that $z \notin F$. In addition to the above, it is obvious that $x \leqslant z$ and $y \leqslant z$ are valid.

Opposite, suppose that the UP filter F in an algebra A satisfies the condition (WIrF) and let S and T ne UP filters in A such that $S \cap T \subseteq F$. Suppose now that $\neg(S \subseteq F)$ and $\neg(T \subseteq F)$ are valid. Then there exists an element $x \in S$ such that $x \notin F$ and there exists an element $y \in T$ such that $y \notin F$. By assumption, there exists a $z \notin F$ such that $x \leqslant z$ and $y \leqslant z$. On the other hand, we have $z \in S$ and $z \in T$ and by consequently $z \in S \cap T \subseteq F$ which is a contradiction. Thus, it must be $S \subseteq F$ or $T \subseteq F$ thus proving that F is a weakly irreducible UP-filter in A. □

The following example illustrates the circumstances in the previous theorem:

Example 4.3. Let A be as in the Examples 3.2. Subset $G := \{0, c\}$ is a weakly irreducible UP-filter in A because for the elements $a, b \notin G$, the following holds $b \notin G$ and $a \leqslant b \wedge b \leqslant b$.

Theorem 4.2. *Every weakly irreducible UP-filter in a meet-commutative UP-algebra is a prime UP-filter.*

Proof. Let F be a weakly irreducible UP-filter in a meet-commutative UP-algebra A and let $x, y \in A$ be such that $x \sqcup y \in F$. If we assume that $x \notin F$ and $y \notin F$, then there exists an element $z \notin F$ such that $x \leqslant z$ and $y \leqslant z$. Since z is the upper bound for x and y, it must be $x \sqcup y \leqslant z$. Then $z \in F$. We got a contradiction. Therefore, it must be $x \in F$ or $y \in F$ which means that F is a prime UP-filter. □

Theorem 4.3. *Every weakly irreducible UP-filter of a meet-commutative UP-algebra is an irreducible UP-filter.*

Proof. The proof of this theorem is directly obtained by combining Theorem 3.1 and Theorem 4.2. □

The proof of Theorem 4.3 can be derived without reference to Theorem 4.2. Indeed:

Proof. Let F be a weakly irreducible UP-filter in A. Let S and T be UP filters in A such that $F = S \cap T$. Then $F \subseteq S$ and $F \subseteq T$. If we assume that $F \neq S$ and $F \neq T$, then there are elements $x \in S$ and $y \in T$ such that $x \notin F$ and $y \notin F$. Since F is a weakly irreducible UP-filter in A, by Theorem 4.1, there is an element $z \notin F$ with $x \leqslant z$ and $y \leqslant z$. On the other hand, we have $z \in S \cap T = F$ which is a contradiction. So, F is irreducible. □

As shown in [8], any prime UP-filter (of the first kind) in a meet-commutative UP-algebra is an irreducible UP-filter. In this paper it is shown that the weakly irreducible UP filter is a prime UP filter (of the first kind) and, therefore, the irreducible UP filter. It is quite natural to ask the question: When will the reverse be true?

One of the answers can be recognized immediately:

Theorem 4.4. *If the lattice $\mathfrak{F}(A)$ of a meet-commutative UP-algebra A is distributive, then any irreducible UP-filter in A is a weakly irreducible UP-filter in A.*

Proof. Let F, S and T be UP-fillers in a meet-commutative UP-algebra A such that $S \cap T \subseteq F$ and F be irreducible. Then

$$F = (S \cap T) \sqcup F = (S \sqcap T) \sqcup F = (S \sqcup F) \sqcap (T \sqcup F) = (S \sqcup F) \cap (T \sqcup F)$$

since $\mathfrak{F}(A)$ is a distributive lattice. As F is an irreducible UP-filter in A, it holds $S \sqcup F = F$ or $T \sqcup F = F$. This implies that $S \subseteq F$ or $T \subseteq F$. Therefore, F is a weakly irreducible UP-filter in A. □

It is quite justified to ask the question: Is this condition necessary?

5. Final comments and possible further work

The algebra class called 'UP-algebra' was designed by A Iampan in [3]. This class of logical algebras has been the subject of many articles (See, for example [4, 5, 11, 15, 17]). As usual, various types of filters in this class of algebra were studied (for example, [4, 5, 11, 14]). One special subclass of these logical algebras, the class of meet-commutative UP-algebras, was introduced by Sawika et al. [15] and somewhat more information on this class of UP-algebra is given by Muhiuddin et al. in the article [8]. In addition, in the last mentioned article, the authors introduced and analyzed the concept of prime UP-filters (of the first kind) and the concept of irreducible UP-filters in meet-commutative UP-algebras. In this paper, as a continuation of previous research, the author introduces and analyzes the concept of weakly irreducible UP-filters in such UP-algebras by weakening the filter's irreducibility requirements. The author is inclined to convey his belief that this research raises a general understanding of meet-commutative algebras to other researchers of the class of UP-algebras.

One of the possible extensions of article [8] and this paper is the analysis of the possibility of introducing the concept of semi-prime UP-filters in meet-commutative UP-algebras:

Let A be a meet-commutative UP-algebra. A UP-filter F of A is said to be a *semi-prime UP-filter* of A if the following holds

$$(\forall x, y, z \in A)((x \sqcup y \in F \wedge x \sqcup z \in F) \implies (\exists d \in A)(d \leqslant y \wedge d \leqslant z)(x \sqcup d \in F)).$$

The term semi-prime ideal was first used by W. Krull in his famous paper [7] (pp. 735). (Cited according to [10], page 107.). Y. Rav adapted this term to general lattices ([10]). In addition, in the mentioned article, Rav introduced and analyzed the concept of semi-prime filters in general lattices, also. These concepts have been studied in several research papers (see, for example, [1]).

6. Acknowledgements

The author is grateful to the reviewer for helpful suggestions that raised the clarity and consistency of the ideas presented in this article.

References

[1] S. Akhter and A. S. A. Noor. Semi prime filters in join semilattices. Annals of Pure and Applied Mathematics, 18(1)(2018), 45–50.
[2] G. Grätzer. General Lattice Theory. Birkhäuser Verlag, 1998.
[3] A. Iampan. A new branch of the logical algebra: UP-algebras. Journal of Algebra and Related Topics, 5(1)(2017), 35–54.
[4] Y. B. Jun and A. Iampan. Implicative UP-filters. Afrika Matematika, 30(7-8)(2019), 1093-1101.
[5] Y. B. Jun and A. Iampan. Comparative and allied UP-filters. Lobachevskii Journal of Mathematics, 40(1)(2019), 60-66.
[6] Y. B. Jun and A. Iampan. Shift UP-filters and decomposition of UP-filters in UP-algebras. Missouri Journal of Mathematical Sciences, 31(1)(2019), 36–45.
[7] W. Krull. Idealtheorie in Ringen ohne Endlichkeitsbedingungen. Mathematicshe Annalen. 101 (1929), 729-744.
[8] G. Muhiuddin, D. A. Romano and Y. B. Jun. Prime and irreducible UP-filters of meet-commutative UP-algebras (Submitted).
[9] C. Prabpayak and U. Leerawat. On ideals and congruences in KU-algebras. Sciencia Magna, 5(1)(2009), 54–57.
[10] Y. Rav. Semiprime ideals in general lattices. Journal of Pure and Applied Algebra, 56(2)(1989), 105–118.
[11] D. A. Romano. Proper UP-filters in UP-algebra. Universal Journal of Mathematics and Applications, 1(2) (2018), 98–100.
[12] D. A. Romano. Two types of prime UP-filters of meet-commutative UP-algebras. Turkish Journal of Mathematics and Computer Science, (Submitted)
[13] D. A. Romano. Prime UP-filter of the third kind in meet-commutative UP-algebras. Annals of Communications in Mathematics, 3(3)(2020), 193–198.
[14] D. A. Romano and Y. B. Jun. Weak implicative UP-filters in UP-algebras. Open Journal of Mathematical Sciences, 4(2020), 442-447.
[15] K. Sawika, R. Intasan, A. Kaewwasri and A. Iampan. Derivation of UP-algebras. Korean Journal of Mathematics, 24(3)(2016), 345–367.
[16] A. Satirad, P. Mosrijai and A. Iampan. Formulas for finding UP-algebras. International Journal of Mathematics and Computer Science, 14(2)(2019), 403-409.
[17] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw and A. Iampan. Fuzzy sets in UP-algebras. Annals of Fuzzy Mathematics and Informatics, 12(2016), 739–756.

International Mathematical Virtual Institute
6, Kordunaška Street, 78000 Banja Luka, Bosnia and Herzegovina
Email address: bato49@hotmail.com