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FUZZY STABILITY RESULTS DERIVING FROM QUADRATIC FUNCTIONAL EQUATION

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ABSTRACT. We establish the generalized Hyers-Ulam stability of the 4 variable quadratic functional equation

$$\phi(2v_1 \pm v_2 \pm v_3 \pm v_4) = \phi(v_1 \pm v_3 \pm v_4) + \phi(v_1 \pm v_2 \pm v_3) + \phi(v_1 \pm v_2 \pm v_4) \\ + \phi(\pm v_1) - \phi(\pm v_2) - \phi(\pm v_3) - \phi(\pm v_4)$$

in Fuzzy Normed Spaces using two different methods.

1. INTRODUCTION AND PRELIMINARIES

Stability of some functional equations within the framework of fuzzy normed spaces or random normed spaces has been investigated. While is it true that a function which about fulfills a functional equation, Hyers theorem become generalized via Aoki for additive mappings and through Rassias for linear mappings with the aid of thinking about an unbounded Cauchy difference. Eventually, the stability problems for several sorts of functional equations in numerous spaces were appreciably studied by using many authors [1, 3, 4, 6, 8, 9, 10, 11, 13, 15, 16, 17, 18].

The well-known functional equation inside the field of stability of functional equation is the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

The function $f(x) = x^2$ is the solution of the functional equation (1.1).

A generalization of the Rassias theorem turned into acquired by means of Gavruta through changing the unbounded difference via a wellknown control function the purpose of this paper is to have a look at the opportunity of changing the most powerful norm within the essential theorem, with an arbitrary continuous norm, so that it will attain a stability end result for fuzzy normed spaces over discipline with valuation Rassias changed into the primary to show that there exists a completely unique linear mapping fulfilling all through the actually field. Numerous outcomes for the Hyers-Ulam-Rassias stability of many functional equations had been proved via numerous researchers [2, 5, 7, 12, 14, 19].

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In this current work, authors establish the Hyers-Ulam stability results of the 4 variable functional equation

$$\phi\left(2v_1 \pm v_2 \pm v_3 \pm v_4\right) = \phi\left(v_1 \pm v_3 \pm v_4\right) + \phi\left(v_1 \pm v_2 \pm v_3\right) + \phi\left(v_1 \pm v_2 \pm v_4\right) + \phi\left(\pm v_1\right) - \phi\left(\pm v_2\right) - \phi\left(\pm v_3\right) - \phi\left(\pm v_4\right) \qquad (1.2)$$

in Fuzzy normed spaces using direct and fixed point methods.

Definition 1.1. Let M be a real linear space. A function $H : M \times R \to [0,1]$ is said to be fuzzy norm on M if for all $v, w \in M$ and $a, b \in R$.

(N_1) $H(v,a) = 0$ for $a \leq 0$;

(N_2) $v = 0$ iff $H(v,a) = 1$ for all $a > 0$;

(N_3) $H(av,b) = H\left(v, \frac{b}{|a|}\right) \quad if \quad a \neq 0$;

(N_4) $H(v+w, a+b) \geq min\{H(v,a), H(w,b)\}$;

(N_5) $H(v,\cdot)$ is a non-decreasing function on R and $\lim_{a\to\infty} H(v,a) = 1$.

(N_6) For $v \neq 0, H(v,\cdot)$ is continuous on R.

The pair (M,H) is called a fuzzy normed linear space.

Theorem 1.1. **[*The Alternative of fixed point*]** *Suppose that for a complete generalized metric space (M,d) and a strictly contractive mapping $T : M \to N$ with Lipschitz constant L. Then, for each given element $v \in M$ either*

(B1) $(T^n v, T^{n+1} v) = +\infty, \quad \forall n \geq 0$, or

(B2) there exists natural number n_0 such that:

i) $d(T^n v, T^{n+1} v) < \infty \quad \forall n \geq n_0$;

ii) The sequence $(T^n v)$ is convergent to a fixed point w^* of T;

iii) w^* is the unique fixed point of T in the set $N = \{w \in M; d(T^{n_0} v, w) < \infty\}$;

iv) $d(w^*, w) \leq \frac{1}{1-L} d(w, Tw) \forall w \in L$.

Throughout the upcoming sections, we consider M, $(Z, H^{'})$ and (N,H) are Linear space, Fuzzy normed space and Fuzzy Banach space respectively. For the notational convenient, we define $\phi : M \to N$ by

$$D\phi(v_1,v_2,v_3,v_4) = \phi\left(2v_1 \pm v_2 \pm v_3 \pm v_4\right) - \phi\left(v_1 \pm v_3 \pm v_4\right) - \phi\left(v_1 \pm v_2 \pm v_3\right) - \phi\left(v_1 \pm v_2 \pm v_4\right) - \phi\left(\pm v_1\right) + \phi\left(\pm v_2\right) + \phi\left(\pm v_3\right) + \phi\left(\pm v_4\right)$$

for all $v_1, v_2, v_3, v_4 \in M$.

2. STABILITY RESULTS FOR (1.2): DIRECT METHOD

Theorem 2.1. *Let $\varphi \in \{-1,1\}$ be fixed and let $\Upsilon : M^4 \to Z$ be a mapping such that for some $\varsigma > 0$ and $\left(\frac{\varsigma}{5^2}\right)^{\varphi} < 1$*

$$H^{'}\left(\Upsilon\left(5^{\varphi}v, 5^{\varphi}v, 5^{\varphi}v, 5^{\varphi}v\right), \tau\right) \geq H^{'}\left(\varsigma^{\varphi}\Upsilon(v,v,v,v), \tau\right) \quad \forall v \in M, \tau > 0 \qquad (2.1)$$

and

$$\lim_{p\to\infty} H^{'}\left(\Upsilon(5^{\varphi p}v_1, 5^{\varphi p}v_2, 5^{\varphi p}v_3, 5^{\varphi p}v_4), 5^{2\varphi p}\tau\right) = 1$$

for all $v_1, v_2, v_3, v_4 \in M$ and all $\tau > 0$. Suppose an even mapping $\phi : M \to N$ satisfies the inequality

$$H\left(D\phi(v_1,v_2,v_3,v_4), \tau\right) \geq H^{'}\left(\Upsilon(v_1,v_2,v_3,v_4), \tau\right) \qquad (2.2)$$

for all $\tau > 0$ and all $v_1, v_2, v_3, v_4 \in M$. Then the limit

$$Q_2(v) = H - \lim_{p\to\infty} \frac{\phi(5^{\varphi p}v)}{5^{2\varphi p}}$$

exists for all $v \in M$ and the mapping $Q_2 : M \to N$ is a unique quadratic mapping such that

$$H\left(\phi(v) - Q_2(v), \tau\right) \geq H^{'}\left(\Upsilon(v, v, v, v), \tau \mid 5^2 - \varsigma \mid\right) \tag{2.3}$$

for all $v \in M$ and all $\tau > 0$.

Proof. Consider $\varphi = 1$. Replacing (v_1, v_2, v_3, v_4) by (v, v, v, v) in (2.2), we get

$$H\left(\phi(5v) - 25\phi(v), \tau\right) \geq H^{'}\left(\Upsilon(v, v, v, v), \tau\right) \quad \forall v \in M, \tau > 0. \tag{2.4}$$

From that (2.4)

$$H\left(\frac{\phi(5v)}{5^2} - \phi(v), \frac{\tau}{5^2}\right) \geq H^{'}\left(\Upsilon(v, v, v, v), \tau\right) \quad \forall v \in M, \tau > 0. \tag{2.5}$$

Interchanging v by $5^p v$ in (2.5), we obtain

$$H\left(\frac{\phi(5^{p+1}v)}{5^2} - \phi(5^p v), \frac{\tau}{5^2}\right) \geq H^{'}\left(\Upsilon(5^p v, 5^p v, 5^p v, 5^p v), \tau\right) \quad \forall v \in M, \tau > 0. \tag{2.6}$$

Using (2.1), (N_3) in (2.6) we get

$$H\left(\frac{\phi(5^{p+1}v)}{5^2} - \phi(5^p v), \frac{\tau}{5^2}\right) \geq H^{'}\left(\Upsilon(v, v, v, v), \frac{\tau}{\varsigma^p}\right) \quad \forall v \in M, \tau > 0. \tag{2.7}$$

It is easy to show that from (2.7), we have

$$H\left(\frac{\phi(5^{p+1}v)}{5^{2(p+1)}} - \frac{\phi(5^p v)}{5^{2p}}, \frac{\tau}{5^{2(p+1)}}\right) \geq H^{'}\left(\Upsilon(v, v, v, v), \frac{\tau}{\varsigma^p}\right) \tag{2.8}$$

holds for all $v \in M, \tau > 0$. Switching τ through $\varsigma^p \tau$ in (2.8), we have

$$H\left(\frac{\phi(5^{p+1}v)}{5^{2(p+1)}} - \frac{\phi(5^p v)}{5^{2p}}, \frac{\varsigma^p \tau}{5^{2(p+1)}}\right) \geq H^{'}\left(\Upsilon(v, v, v, v), \tau\right) \quad \forall v \in M, \tau > 0. \tag{2.9}$$

It is easy to show that

$$\frac{\phi(5^p v)}{5^{2p}} - \phi(v) = \sum_{l=0}^{p-1} \frac{\phi(5^{l+1}v)}{5^{2(l+1)}} - \frac{\phi(5^l v)}{5^{2l}} \tag{2.10}$$

for all $v \in M$. From (2.9) and (2.10), we have

$$\begin{aligned} H\left(\frac{\phi(5^p v)}{5^{2p}} - \phi(v), \sum_{l=0}^{p-1} \frac{\tau\varsigma^l}{5^{2(l+1)}}\right) &\geq \min\{H\left(\frac{\phi(5^{l+1}v)}{5^{2(l+1)}} - \frac{\phi(5^l v)}{5^{2l}}, \frac{\tau\varsigma^l}{5^{2(l+1)}}\right) \ : l = 0, 1, 2, 3\} \\ &\geq H^{'}\left(\Upsilon\left(v, v, v, v\right), \tau\right) \quad \forall v \in M, \tau > 0. \end{aligned} \tag{2.11}$$

Interchanging v by $5^q v$ in (2.11) and utilizing (2.1), (N_3), we reach

$$H\left(\frac{\phi(5^{p+q}v)}{5^{2(p+q)}} - \frac{\phi(5^q v)}{5^{2q}}, \sum_{l=0}^{p-1} \frac{\tau\varsigma^l}{5^{2(l+1)}}\right) \geq H^{'}\left(\Upsilon(5^q v, 5^q v, 5^q v, 5^q v), \tau\right)$$

$$\geq H^{'}\left(\Upsilon(v,v,v,v),\tfrac{\tau}{\varsigma^q}\right)$$

and so

$$H\left(\frac{\phi(5^{p+q}v)}{5^{2(p+q)}}-\frac{\phi(5^q v)}{5^{2q}},\sum_{l=q}^{p+q-1}\frac{\tau\varsigma^l}{5^{2(l+1)}}\right)\geq H^{'}\left(\Upsilon(v,v,v,v),\tau\right)\quad\forall v\in M,\tau>0. \tag{2.12}$$

and all $p,q\geq 0$. Replacing τ by $\frac{\tau}{\sum_{l=q}^{p+q-1}\frac{\varsigma^l}{5^{2(l+1)}}}$ in (2.12), we get

$$H\left(\frac{\phi(5^{p+q}v)}{5^{2(p+q)}}-\frac{\phi(5^q v)}{5^{2q}},\tau\right)\geq H^{'}\left(\Upsilon(v,v,v,v),\frac{\tau}{\sum_{l=q}^{p+q-1}\frac{\varsigma^l}{5^{2(l+1)}}}\right)\quad\forall v\in M,\tau>0. \tag{2.13}$$

and all $p,q\geq 0$. As $0<\varsigma<5^2$ and $\sum_{l=0}^{p}\left(\frac{\varsigma}{5^2}\right)^l<\infty$, the Cauchy criterion for convergence and (N_5) implies that $\{\frac{\phi(5^p v)}{5^{2p}}\}$ is a Cauchy sequence in (N,H). Since (N,H) is a fuzzy Banach space, this $\{\frac{\phi(5^p v)}{5^{2p}}\}$ converges to some point $Q_2(v)\in N$. So one can define $Q_2:M\to N$ by

$$Q_2(v):=H-\lim_{p\to\infty}\frac{\phi(5^p v)}{5^{2p}}$$

for all $v\in M$. Since ϕ is even. Letting $p=0$ in (2.13), we obtain

$$H\left(\frac{\phi(5^p v)}{5^{2p}}-\phi(v),\tau\right)\geq H^{'}\left(\Upsilon(v,v,v,v),\frac{\tau}{\sum_{l=0}^{p-1}\frac{\varsigma^l}{5^{2(l+1)}}}\right)\quad\forall v\in M,\tau>0. \tag{2.14}$$

Passing the limit as $p\to\infty$ in (2.14) and utilizing (N_6), we have

$$H\left(\phi(v)-Q_2(v),\tau\right)\geq H^{'}\left(\Upsilon(v,v,v,v),\tau(5^2-\varsigma)\right)$$

for all $v\in M$ and all $\tau>0$. Now we claim that Q_2 is quadratic. Replacing (v_1,v_2,v_3,v_4) by $(5^p v_1,5^p v_2,5^p v_3,5^p v_4)$ in (2.2) respectively, we have

$$H\left(\frac{1}{5^{2p}}D\phi\left(5^p v_1,5^p v_2,5^p v_3,5^p v_4\right),\tau\right)\geq H^{'}\left(\Upsilon(5^p v_1,5^p v_2,5^p v_3,5^p v_4),5^{2p}\tau\right)$$

for all $v\in M$ and all $\tau>0$. Since

$$\lim_{p\to\infty}H^{'}\left(\Upsilon(5^p v_1,5^p v_2,5^p v_3,5^p v_4),5^{2p}\tau\right)=1$$

Q_2 fulfils (1.2). Therefore, $Q_2:M\to N$ is quadratic. Next, to show the uniqueness of Q_2, let R_2 be another quadratic function which fulfilling (2.3). Fix $v\in M$, clearly $Q_2(5^p v)=5^{2p}Q_2(v)$ and $R_2(5^p v)=5^{2p}R_2(v)\quad\forall v\in M,\ p>0$. From (2.3), we have

$$\begin{aligned}H(Q_2(v)-R_2(v),\tau)\qquad&=H\left(\frac{Q_2(5^p v)}{5^{2p}}-\frac{R_2(5^p v)}{5^{2p}},\tau\right)\\&\geq H^{'}\left(\Upsilon(v,v,v,v),\frac{(5^{2p})\tau(5^2-\varsigma)}{2\varsigma^p}\right)\end{aligned}$$

for all $v\in M$ and all $\tau>0$. Since $\lim_{p\to\infty}\frac{(5^{2p})\tau(5^2-\varsigma)}{2\varsigma^p}=\infty$, we have

$$\lim_{p\to\infty}H^{'}\left(\Upsilon(v,v,v,v),\frac{(5^{2p})\tau(5^2-\varsigma)}{2\varsigma^p}\right)=1.$$

Thus $H(Q_2(v)-R_2(v),\tau)=1$ for all $v\in M$ and all $\tau>0$, and so $Q_2(v)=R_2(v)$. In similar manner, we can obtain the stability results for $\varphi=-1$. $\square$

Corollary 2.2. *Suppose that the function $\phi : M \to N$ satisfies the inequality*

$$H(D\phi(v_1,v_2,v_3,v_4),\tau) \geq \begin{cases} H'(\eta,\tau), \\ H'\left(\eta||v_1||^a + ||v_2||^a + ||v_3||^a + ||v_4||^a, \tau\right), \\ H'\Big(\eta\Big(||v_1||^a + ||v_2||^a + ||v_3||^a + ||v_4||^a + \\ \qquad\qquad ||v_1||^{4a}\cdot||v_2||^{4a}\cdot||v_3||^{4a}\cdot||v_4||^{4a}\Big),\tau\Big), \end{cases}$$

for all $v_1, v_2, v_3, v_4 \in M$ and all $\tau > 0$, where η, a are constants with $\eta > 0$. Then there exists a unique quadratic mapping $Q_2 : M \to N$ such that

$$H(\phi(v) - Q_2(v),\tau) \geq \begin{cases} H'(\eta, \mid 24 \mid \tau), \\ H'\left(4\eta||v||^a, \mid 5^2 - 5^a \mid \tau\right); & a \neq 2, \\ H'\left(5\eta||v||^{4a}, \mid 5^2 - 5^{4a} \mid \tau\right); & a \neq \frac{2}{4}, \end{cases}$$

for all $v \in M$ and all $\tau > 0$.

3. Stability results for (1.2):fixed point method

For to prove the stability result, we define the following θ_l is a constant such that

$$\theta_l = \begin{cases} 5 & \text{if } \quad l = 0; \\ \frac{1}{5} & \text{if } \quad l = 1; \end{cases}$$

and χ is the set such that $\chi = \{s/s : M \to N, s(0) = 0\}$.

Theorem 3.1. *Let $\phi : M \to N$ be a mapping for which there exists a function $\Upsilon : M^4 \to N$ with condition*

$$\lim_{p\to\infty} H^{'}\left(\Upsilon(\theta^p v_1, \theta^p v_2, \theta^p v_3, \theta^p v_4), \theta^{2p}\tau\right) = 1 \quad \forall v_1,v_2,v_3,v_4 \in M,\ \tau > 0, \tag{3.1}$$

and fulfilling the inequality

$$H(D\phi(v_1,v_2,v_3,v_4),\tau) \geq H^{'}(\Upsilon(v_1,v_2,v_3,v_4),\tau) \quad \forall v_1,v_2,v_3,v_4 \in M,\ \tau > 0. \tag{3.2}$$

If there exist $L = L[l]$ such that $v \to \gamma(v) = \Upsilon\left(\frac{v}{5},\frac{v}{5},\frac{v}{5},\frac{v}{5}\right)$ has the property

$$H^{'}\left(L\frac{1}{\theta_l^2}\gamma(\theta_l v),\tau\right) = H^{'}\left(\gamma(v),\tau\right) \quad \forall v \in M,\ \tau > 0, \tag{3.3}$$

then there exist unique quadratic function $Q_2 : M \to N$ fulfilling the functional equation (1.2) and

$$H(\phi(v) - Q_2(v),\tau) \geq H^{'}\left(\frac{L^{1-l}}{1-L}\gamma(v),\tau\right) \quad \forall v \in M,\ \tau > 0.$$

Proof. Let d be a general metric on χ such that

$$d(t,s) = inf\left\{p \in (0,\infty) | H(t(v) - s(v),\tau) \geq H^{'}(\gamma(v),p\tau), v \in M, \tau > 0\right\}.$$

It is easy to see that (χ, d) is complete. Define $T : \chi \to \chi$ by $Tt(v) = \frac{1}{\theta_l^2}t(\theta_l v) \quad \forall v \in M$, for $t, s \in \chi$, we obtain

$$\begin{aligned} d(t,s) = p &\Rightarrow H\left(t(v) - s(v),\tau\right) \geq H^{'}(\gamma(v),p\tau) \\ &\Rightarrow H\left(\frac{t(\theta_l v)}{\theta_l^2} - \frac{s(\theta_l v)}{\theta_l^2},\tau\right) \geq H^{'}(\gamma(\theta_l v),p\theta_l^2\tau) \\ &\Rightarrow H(Tt(v) - Ts(v),\tau) \geq H^{'}(\gamma(\theta_l v),p\theta_l^2\tau) \end{aligned} \tag{3.4}$$

$$\Rightarrow H(Tt(v) - Ts(v), \tau) \geq H^{'}(\gamma(v), pL\tau)$$
$$\Rightarrow d(Tt(v), Ts(v)) \geq pL$$
$$\Rightarrow d(Tt, Ts, \tau) \geq Ld(t, s) \quad \forall\, t, s \in \chi.$$

So, T is strictly contractive mapping on χ with Lipschitz constant L, switching (v_1, v_2, v_3, v_4) by (v, v, v, v) in (3.2), we have

$$H\left(\phi(5v) - 25\phi(v), \tau\right) \geq H^{'}\left(\Upsilon(v, v, v, v), \tau\right) \quad \forall v \in M,\ \tau > 0. \tag{3.5}$$

Using (N_3) in (3.5), we reach

$$H\left(\frac{\phi(5v)}{5^2} - \phi(v), \tau\right) \geq H^{'}\left(\frac{\Upsilon(v, v, v, v)}{5^2}, \tau\right) \quad \forall v \in M,\ \tau > 0, \tag{3.6}$$

with the help of (3.3) when $l = 0$, it follows from (3.6) that

$$\Rightarrow H\left(\frac{\phi(5v)}{5^2} - \phi(v), \tau\right) \geq H^{'}(L\gamma(v), \tau)$$
$$\Rightarrow d(T\phi, \phi) \leq L = L^1 = L^{1-l}. \tag{3.7}$$

Replacing v by $\frac{v}{5}$ in (3.5), we get

$$H\left(\phi(v) - 25\phi\left(\frac{v}{5}\right), \tau\right) \geq H^{'}\left(\Upsilon\left(\frac{v}{5}, \frac{v}{5}, \frac{v}{5}, \frac{v}{5}\right), \tau\right) \quad \forall v \in M,\ \tau > 0,$$

when $l = 1$, it follows from (3.7), we reach

$$\Rightarrow H\left(\phi(v) - 25\phi\left(\frac{v}{5}\right), \tau\right) \geq H^{'}(\gamma(v), \tau)$$
$$\Rightarrow d(\phi, T\phi) \leq 1 = L^0 = L^{1-l}. \tag{3.8}$$

Then from (3.7) and (3.8), we can conclude

$$\Rightarrow d(\phi, T\phi) \leq L^{1-l} < \infty.$$

Now, from Theorem 1.1 in both cases, it follows that there exists a fixed point Q_2 of T in χ such that

$$Q_2(v) = H - \lim_{p \to \infty} \frac{\phi(\theta^p v)}{\theta^{2p}} \quad \forall\, v \in M \text{ and } \tau > 0.$$

Replacing $(v_1, v_2, v_{3,}, v_4)$ by $(\theta_l^p v_1, \theta_l^p v_2, \theta_l^p v_3, \theta_l^p v_4)$ in (3.2), we arrive

$$H\left(\frac{1}{\theta_l^{2p}} D\phi(\theta_l^p v_1, \theta_l^p v_2, \theta_l^p v_3, \theta_l^p v_4), \tau\right) \geq H^{'}\left(\Upsilon(\theta_l^p v_1, \theta_l^p v_2, \theta_l^p v_3, \theta_l^p v_4), \theta_l^{2p}\tau\right)$$

for all $\tau > 0$ and all $v_1, v_2, v_3, v_4 \in M$. By similar procedure of Theorem 2.1, we show that the function $Q_2 : M \to N$ is quadratic and it fulfils (1.2). By Theorem 1.1, as Q_2 is unique fixed point of T in the set

$$\Delta = \left\{\phi \in \chi / d(\phi, Q_2) < \infty\right\}.$$

So that, Q_2 is a unique function such that

$$H(\phi(v) - Q_2(v), \tau) \geq H^{'}(\gamma(v), p\tau) \quad \forall v \in M,\ \tau > 0.$$

Again utilizing Theorem 1.1, we reach

$$d(\phi, Q_2) \leq \frac{1}{1 - L} d(\phi, T\phi)$$
$$\Rightarrow d(\phi, Q_2) \leq \frac{L^{1-l}}{1 - L}$$

$$\Rightarrow H(\phi(v) - Q_2(v), \tau) \geq H^{'}\left(\gamma(v)\frac{L^{1-l}}{1-L}, \tau\right) \quad \forall v \in M,\ \tau > 0.$$

□

Corollary 3.2. *Suppose a function $\phi : M \to N$ fulfils the inequality*

$$H(D\phi(v_1, v_2, v_3, v_4), \tau) \geq \begin{cases} H'(\eta, \tau), \\ H'\left(\eta||v_1||^a + ||v_2||^a + ||v_3||^a + ||v_4||^a, \tau\right), \\ H'\Big(\eta\Big(||v_1||^a + ||v_2||^a + ||v_3||^a + ||v_4||^a + \\ \qquad\qquad ||v_1||^{4a} \cdot ||v_2||^{4a} \cdot ||v_3||^{4a} \cdot ||v_4||^{4a}\Big), \tau\Big), \end{cases}$$

for all $v_1, v_2, v_3, v_4 \in M$ and $\tau > 0$, where η, a are constants with $\eta > 0$. Then there exists a unique quadratic mapping $Q_2 : M \to N$ such that

$$H(\phi(v) - Q_2(v), \tau) \geq \begin{cases} H^{'}\left(\eta, \frac{\tau}{|24|}\right), \\ H^{'}\left(4\eta||v||^a, \tau \mid 5^2 - 5^a \mid\right); & a \neq 2, \\ H^{'}\left(5\eta||v||^{4a}, \tau \mid 5^2 - 5^{4a} \mid\right); & a \neq \frac{2}{4}. \end{cases}$$

for all $v \in M$ and $\tau > 0$.

Proof. Setting

$$\Upsilon(v_1, v_2, v_3, v_4) \leq \begin{cases} \eta, \\ \eta(\sum_{m=1}^4 ||v_m||^a), \\ \eta(\prod_{m=1}^4 ||v_m||^a + \sum_{m=1}^4 ||v_m||^{4a}), \end{cases}$$

for all $v_1, v_2, v_3, v_4 \in M$. Then

$$H^{'}\left(\Upsilon\left(\theta_l^p v_1, \theta_l^p v_2, \theta_l^p v_3, \theta_l^p v_4\right), \theta_l^{2p}\tau\right) = \begin{cases} H^{'}(\eta, \theta_l^{2p}\tau), \\ H^{'}\left(\eta\sum_{m=1}^4 ||v_m||^a, \theta_l^{(2-a)p}\tau\right), \\ H^{'}\left(\eta(\sum_{m=1}^4 ||v_l||^{4a} + \prod_{m=1}^4 ||v_m||^a), \theta_m^{(2-4a)p}\tau\right). \end{cases}$$

$$= \begin{cases} \to & 1 & as & p \to \infty, \\ \to & 1 & as & p \to \infty, \\ \to & 1 & as & p \to \infty. \end{cases}$$

Thus, (2.1) is holds. But we get

$$\gamma(v) = \Upsilon\left(\frac{v}{5}, \frac{v}{5}, \frac{v}{5}, \frac{v}{5}\right)$$

has the property

$$H^{'}\left(L\frac{1}{\theta_l^2}\gamma(\theta_l v), \tau\right) \geq H^{'}(\gamma(v), \tau) \quad \forall v \in M,\ \tau > 0.$$

Hence,

$$H^{'}(\gamma(v), \tau) = H^{'}\left(\Upsilon\left(\frac{v}{5}, \frac{v}{5}, \frac{v}{5}, \frac{v}{5}\right), \tau\right)$$
$$= \begin{cases} H^{'}(\eta, \tau), \\ H^{'}\left(\frac{4}{5^a}\eta||v||^a, \tau\right), \\ H^{'}\left(\frac{5}{5^{4a}}\eta||v||^{4a}, \tau\right). \end{cases}$$

Now,

$$H^{'}\left(\frac{1}{\theta_l^2}\gamma(\theta_l v),\tau\right) = \begin{cases} H^{'}\left(\frac{\eta}{\theta_l^2},\tau\right), \\ H^{'}\left(\frac{\eta}{\theta_l^2}\left(\frac{4}{5^a}\right)||\theta_l v||^a,\tau\right), \\ H^{'}\left(\frac{\eta}{\theta_l^2}\left(\frac{5}{5^{4a}}\right)||\theta_l v||^{4a},\tau\right). \end{cases}$$

$$= \begin{cases} H^{'}(\theta_l^{-2}\gamma(v),\tau), \\ H^{'}(\theta_l^{a-2}\gamma(v),\tau), \\ H^{'}(\theta_l^{4a-2}\gamma(v),\tau). \end{cases}$$

Next, from the following cases for the conditions (i) and (ii).

Case(i):$L = 5^{-2} \quad for \quad a = 0 \quad if \quad l = 0.$

$$H(\phi(v) - Q_2(v),\tau) \geq H^{'}\left(\tfrac{L^{1-l}}{1-L}\gamma(v),\tau\right) \geq H^{'}\left(\tfrac{5^{-2}}{1-5^{-2}}\eta,\tau\right) \geq H^{'}(\eta, 24\tau).$$

Case(ii):$L = \left(\frac{1}{5}\right)^{-2} \quad for \quad a = 0 \quad if \quad l = 1.$

$$H(\phi(v) - Q_2(v),\tau) \geq H^{'}\left(\tfrac{L^{1-l}}{1-L}\gamma(v),\tau\right) \geq H^{'}\left(\frac{1}{1-\left(\frac{1}{5}\right)^{-2}}\eta,\tau\right) \geq H^{'}(\eta, -24\tau).$$

Case(iii):$L = (5)^{a-2} \quad for \quad a < 2 \quad if \quad l = 0.$

$$\begin{aligned} H(\phi(v) - Q_2(v),\tau) \geq H^{'}\left(\frac{L^{1-l}}{1-L}\gamma(v),\tau\right) &\geq H^{'}\left(\frac{5^{a-2}}{1-5^{a-2}}\frac{4\eta||v||^a}{5^a},\tau\right) \\ &\geq H^{'}\left(4\eta||v||^a, \tau(5^2 - 5^a)\right). \end{aligned}$$

Case(iv):$L = (5)^{2-a} \quad for \quad a > 2 \quad if \quad l = 1.$

$$\begin{aligned} H(\phi(v) - Q_2(v),\tau) \geq H^{'}\left(\frac{L^{1-l}}{1-L}\gamma(v),\tau\right) &\geq H^{'}\left(\frac{5^{2-a}}{1-5^{2-a}}\frac{4\eta||v||^a}{5^a},\tau\right) \\ &\geq H^{'}\left(4\eta||v||^a, \tau(5^a - 5^2)\right). \end{aligned}$$

Case(v):$L = (5)^{4a-2} \quad for \quad a < \frac{1}{4} \quad if \quad l = 0\,.$

$$\begin{aligned} H(\phi(v) - Q_2(v),\tau) \geq H^{'}\left(\frac{L^{1-l}}{1-L}\gamma(v),\tau\right) &\geq H^{'}\left(\frac{5^{4a-2}}{1-5^{4a-2}}\frac{5\eta||v||^{4a}}{5^{4a}},\tau\right) \\ &\geq H^{'}\left(5\eta||v||^{4a}, \tau(5^2 - 5^{4a})\right). \end{aligned}$$

Case(vi):$L = (5)^{2-4a} \quad for \quad a > \frac{1}{4} \quad if \quad l = 1.$

$$\begin{aligned} H(\phi(v) - Q_2(v),\tau) \geq H^{'}\left(\frac{L^{1-l}}{1-L}\gamma(v),\tau\right) &\geq H^{'}\left(\frac{5^{2-4a}}{1-5^{2-4a}}\frac{5\eta||v||^{4a}}{5^{4a}},\tau\right) \\ &\geq H^{'}\left(5\eta||v||^{4a}, \tau(5^{4a} - 5^2)\right). \end{aligned}$$

□

4. CONCLUSION

In this work, we investigated Hyers-Ulam stability results for the quadratic functional equation (1.2). In section 2, we examined Hyers-Ulam stability results of the quadratic functional equation (1.2) in fuzzy normed space by utilizing direct method. In section 3, we obtained Hyers-Ulam stability results of the quadratic functional equation (1.2) in fuzzy normed space by utilizing fixed point method.

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