



POSITIVE SOLUTIONS TO A DERIVATIVE DEPENDENT p -LAPLACIAN EQUATION WITH RIEMANN-STIELTJES INTEGRAL BOUNDARY CONDITIONS

SESHADEV PADHI, JAFFAR ALI AND JOHN R. GRAEF*

ABSTRACT. This paper is concerned with the existence of two nontrivial positive solutions to a class of boundary value problems involving a p -Laplacian of the form

$$\begin{aligned} (\Phi_p(x'))' + g(t)f(t, x, x') &= 0, \quad t \in (0, 1), \\ x(0) - ax'(0) &= \alpha[x], \\ x(1) + bx'(1) &= \beta[x], \end{aligned}$$

where $\Phi_p(x) = |x|^{p-2}x$ is a one dimensional p -Laplacian operator with $p > 1$, a and b are real constants, and α and β are given by the Riemann-Stieltjes integrals

$$\alpha[x] = \int_0^1 x(t)dA(t), \quad \beta[x] = \int_0^1 x(t)dB(t),$$

with A and B functions of bounded variation. The approach used is based on fixed point index theory. The results obtained in this paper are new in the literature.

1. INTRODUCTION

In this paper, we discuss the existence of positive solutions to the nonlinear boundary value problem (BVP) with p -Laplacian

$$(\Phi_p(x'))' + g(t)f(t, x, x') = 0, \quad t \in (0, 1), \tag{1.1}$$

together with the non-local boundary conditions

$$\begin{aligned} x(0) - ax'(0) &= \alpha[x], \\ x(1) + bx'(1) &= \beta[x], \end{aligned} \tag{1.2}$$

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*Corresponding author.

where a and b are constants, α and β are linear functionals on $C[0, 1]$ defined by the Riemann-Stieltjes integrals

$$\alpha[x] = \int_0^1 x(t)dA(t), \quad \beta[x] = \int_0^1 x(t)dB(t), \quad (1.3)$$

A and B are functions of bounded variation, not both of which are identically zero. Here, dA and dB can be signed measures. In (1.1), the function $\Phi_p(x) = |x|^{p-2}x$ is a one-dimensional p -Laplacian operator with $p > 1$, and the inverse operator Φ_q is defined by $\Phi_p^{-1}(x) = \Phi_q(x) = |x|^{q-2}x$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Riemann-Stieltjes integrals play an important role in the literature, and as given in (1.3), they include a variety of non-local boundary conditions such as:

$$\alpha[x] = \lambda x(\eta), \quad \lambda \geq 0, \quad \eta \in (0, 1);$$

$$\alpha[x] = \sum_{j=1}^l \lambda_j x(\mu_j), \quad \lambda_i \in \mathbb{R}, \quad j = 1, 2, \dots, l, \quad 0 < \eta_1 < \eta_2 < \dots < \eta_l < 1;$$

$$\alpha[x] = \int_0^1 x(t)h(t)dt, \quad h \in C((0, 1), \mathbb{R}).$$

If $p = 2$, then (1.1) reduces to the second order ordinary differential equation

$$x'' + g(t)f(t, x, x') = 0, \quad t \in (0, 1). \quad (1.4)$$

Equation (1.4) together with the boundary conditions (1.2) has been studied by many authors in the literature, for example, see [2, 7, 12, 13, 16]. In a recent paper, Yang and Wang [14] used the Avery-Peterson fixed point theorem to study the existence of at least three positive solutions to the p -Laplacian equation (1.1) together with integral boundary conditions of the type

$$x(0) - ax'(0) = \int_0^1 g_1(s)x(s)ds, \quad (1.5a)$$

$$x(1) + bx'(1) = \int_0^1 g_2(s)x(s)ds, \quad (1.5b)$$

where $a, b \geq 0$ and $p > 1$. For BVPs with a p -Laplacian, we may refer the reader to [1, 3, 5, 9, 10, 11, 15, 17, 18] and the references cited therein. The main tools used in the above-cited papers are the upper-lower solution method, Krasnosel'skii's fixed point theorem, the Avery-Peterson fixed point theorem, the Leggett-William fixed point theorem, and fixed point index theory. Webb [12] and Webb and Infante [13] proved that the fixed point index approach is one of the most efficient methods to study the existence of multiple positive solutions of the problem (1.4) together with the boundary condition (1.2).

We note that the integrals on the right hand side of (1.5) are particular cases of the Riemann-Stieltjes integrals $\alpha[x]$ and $\beta[x]$ defined in (1.3). There does not appear to be any results in the literature on the existence of positive solutions of BVP (1.1)–(1.2). In this paper, we shall use the method adopted in [14] to obtain an equivalent integral equation, and then use the fixed point index approach to study (1.1)–(1.2).

In order to obtain our existence results, we shall use the following hypotheses throughout this paper.

- (A1) $f \in C([0, 1] \times [0, \infty) \times \mathbb{R}, [0, \infty))$, $g \in C([0, 1], [0, \infty))$, and g does not vanish identically on any subinterval of $[0, \infty)$;
- (A2) $0 < \alpha[1] < 1$ and $0 < \beta[1] < 1$;

(A3) There exists a constant $\mu > 0$ and a continuous function $p_f : (0, 1) \rightarrow [0, \infty)$ such that

$$f(t, x, x') \leq p_f(t) \quad \text{for } 0 \leq t \leq 1, \mu \leq x < \infty, -\infty < x' < \infty;$$

(A4) We have

$$\Phi_q \left(\int_0^1 g(r)p_f(r)dr \right) < \infty,$$

$$\frac{a\Phi_q \left(\int_0^\rho g(r)p_f(r)dr \right) + \int_0^1 \int_0^s \Phi_q \left(\int_\theta^\rho g(r)p_f(r)dr \right) d\theta dA(s)}{1 - \alpha[1]} + \int_0^\rho \Phi_q \left(\int_s^\rho g(r)p_f(r)dr \right) ds < \infty$$

for $s \leq \rho$, and

$$\frac{b\Phi_q \left(\int_\rho^1 g(r)p_f(r)dr \right) + \int_0^1 \int_s^1 \Phi_q \left(\int_\rho^\theta g(r)p_f(r)dr \right) d\theta dB(s)}{1 - \beta[1]} + \int_\rho^1 \Phi_q \left(\int_\rho^s g(r)p_f(r)dr \right) ds < \infty$$

for $s \geq \rho$.

$$(A5) \quad \int_0^1 \int_0^t \Phi_q \left(\int_s^\rho g(r)p_f(r)dr \right) ds dB(t) \geq 0 \quad \text{and} \\ \int_0^1 \int_t^1 \Phi_q \left(\int_\rho^s g(r)p_f(r)dr \right) ds dA(t) \geq 0.$$

Throughout this work, we consider the Banach space $X = C^1([0, 1])$ equipped with the norm

$$\|x\|_{C^1} = \max\{\|x\|_\infty, \|x'\|_\infty\},$$

where

$$\|v\|_\infty = \max_{0 \leq t \leq 1} |v(t)|.$$

Section 2 in this paper contains some basic results for our problem. Our main results and their proofs are in Section 3, and the last section contains some further discussion and some examples to illustrate the applicability of our results.

2. PRELIMINARIES

In this section, we provide results similar to the those obtained in [14]. The proofs of our Lemmas 2.1–2.5 are similar to the those of Lemmas 2.1–2.5 in [14]. Lemmas 2.1–2.5 in [14] are similar to lemmas in [8] and [15]. However, for completeness sake, we do include proofs of our lemmas.

Lemma 2.1. *Assume that*

$$1 - \alpha[1] \neq 0 \quad \text{and} \quad 1 - \beta[1] \neq 0. \quad (2.1)$$

Then for any given $y \in X$, the BVP

$$-(\Phi_p(x'))' = y(t) \text{ for a.e. } t \in (0, 1), \quad (2.2)$$

$$x(0) - \alpha x'(0) = \alpha[x], \quad (2.3)$$

$$x(1) + bx'(1) = \beta[x],$$

has solutions given by

$$x(t) = \frac{a\Phi_q(\bar{\phi}_0) + \int_0^1 \int_0^t \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds dA(t)}{1 - \alpha[1]} + \int_0^t \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds \quad (2.4)$$

and

$$x(t) = - \frac{b\Phi_q \left(\bar{\phi}_0 - \int_0^1 y(r) dr \right) + \int_0^1 \int_t^1 \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds dB(t)}{1 - \beta[1]} - \int_t^1 \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds, \quad (2.5)$$

where $\bar{\phi}_0$ satisfies the integral equation

$$a\Phi_q(\bar{\phi}_0) = \int_0^1 \int_t^1 \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds dA(t) - \int_0^1 \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds - \left(\frac{1 - \alpha[1]}{1 - \beta[1]} \right) \left[b\Phi_q \left(\bar{\phi}_0 - \int_0^1 y(r) dr \right) + \int_0^1 \int_t^1 \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds dB(t) \right]. \quad (2.6)$$

Proof. Integrating (2.2) from 0 to t , we have

$$x'(t) = \Phi_q \left(\bar{\phi}_0 - \int_0^t y(s) ds \right), \quad (2.7)$$

where $\bar{\phi}_0 = \Phi_p(x'(0))$. Integrating (2.7) from t to 1, we obtain

$$x(t) = x(1) - \int_t^1 \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds. \quad (2.8)$$

Substituting the second boundary condition in (2.3) into (2.8) gives

$$x(t) = -bx'(1) + \beta[x] - \int_t^1 \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds. \quad (2.9)$$

Multiplying both sides of (2.9) by $dB(t)$ and integrating from 0 to 1, we obtain

$$\beta[x] = \frac{bx'(1)\beta[1] + \int_0^1 \int_t^1 \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds dB(t)}{\beta[1] - 1}. \quad (2.10)$$

Using the above value of $\beta[x]$ in (2.9), we obtain

$$x(t) = -\frac{bx'(1) + \int_0^1 \int_t^1 \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds dB(t)}{1 - \beta[1]} - \int_t^1 \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds. \quad (2.11)$$

Using (2.7) with $t = 1$ in (2.11), we obtain (2.5).

Next, we integrate (2.7) from 0 to t to obtain

$$x(t) = x(0) + \int_0^t \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds. \quad (2.12)$$

Multiplying both sides of (2.12) by $dA(t)$, integrating from 0 to 1, and using the first boundary condition in (2.3), we have

$$\alpha[x] = \frac{ax'(0)\alpha[1] + \int_0^1 \int_0^t \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds dA(t)}{1 - \alpha[1]}. \quad (2.13)$$

Since $x'(0) = \Phi_q(\bar{\phi}_0)$ (from (2.7)), using the above value of $\alpha[x]$ in the boundary condition $x(0) - ax'(0) = \alpha[x]$, we obtain

$$x(0) = \frac{a\Phi_q(\bar{\phi}_0) + \int_0^1 \int_0^t \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds dA(t)}{1 - \alpha[1]}$$

Using the above value of $x(0)$ in (2.12) yields (2.4).

Finally, since

$$x'(1) = \Phi_q \left(\bar{\phi}_0 - \int_0^1 y(r) dr \right), \quad (2.14)$$

from (2.10), we obtain

$$\beta[x] = -\frac{b\beta[1]\Phi_q \left(\bar{\phi}_0 - \int_0^1 y(r) dr \right) + \int_0^1 \int_0^1 \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds dB(t)}{1 - \beta[1]} \quad (2.15)$$

By (2.4), we have

$$x(1) = \frac{a\Phi_q(\bar{\phi}_0) + \int_0^1 \int_0^t \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds dA(t)}{1 - \alpha[1]} + \int_0^1 \Phi_q \left(\bar{\phi}_0 - \int_0^s y(r) dr \right) ds \quad (2.16)$$

Using (2.14)–(2.16) in the boundary condition $x(1) + bx'(1) = \beta[x]$, we obtain (2.6). This completes the proof of the lemma. \square

Lemma 2.2. *Let (2.1) hold and $y \in C[0,1]$ with $y \geq 0$. Then there exist constants $l \in \left(0, \int_0^1 y(s) ds\right)$ and $\rho \in (0,1)$ such that (2.6) is satisfied for $\tilde{\phi}_0 = l := \int_0^\rho y(r) dr$.*

Proof. For any $\theta \in [0, \infty)$, set

$$\begin{aligned} \lambda(\theta) = & a\Phi_q(\theta) - \int_0^1 \int_t^1 \Phi_q \left(\theta - \int_0^s y(r) dr \right) ds dA(t) + \int_0^1 \Phi_q \left(\theta - \int_0^s y(r) dr \right) ds \\ & + \frac{(1 - \alpha[1])}{(1 - \beta[1])} \left[b\Phi_q \left(\theta - \int_0^1 y(r) dr \right) + \int_0^1 \int_t^1 \Phi_q \left(\theta - \int_0^s y(r) dr \right) ds dB(t) \right]; \end{aligned}$$

then $\lambda(\theta)$ can be rewritten as

$$\begin{aligned} \lambda(\theta) = & a\Phi_q(\theta) + \int_0^1 \int_0^t \Phi_q \left(\theta - \int_0^s y(r) dr \right) ds dA(t) \\ & + (1 - \alpha[1]) \int_0^1 \Phi_q \left(\theta - \int_0^s y(r) dr \right) ds \\ & + \frac{(1 - \alpha[1])}{(1 - \beta[1])} \left[b\Phi_q \left(\theta - \int_0^1 y(r) dr \right) + \int_0^1 \int_t^1 \Phi_q \left(\theta - \int_0^s y(r) dr \right) ds dB(t) \right]. \end{aligned}$$

Notice that $\lambda(\theta)$ is continuous on $[0, \infty)$ by the continuity of Φ_q and y . Furthermore, $\lambda(0) < 0$ and

$$\begin{aligned} \lambda \left(\int_0^1 y(r) dr \right) = & a\Phi_q \left(\int_0^1 y(r) dr \right) + \int_0^1 \int_0^t \Phi_q \left(\int_s^1 y(r) dr \right) ds dA(t) \\ & + (1 - \alpha[1]) \int_0^1 \Phi_q \left(\int_s^1 y(r) dr \right) ds \\ & + \frac{(1 - \alpha[1])}{(1 - \beta[1])} \left[b\Phi_q(0) + \int_0^1 \int_t^1 \Phi_q \left(\int_s^1 y(r) dr \right) ds dB(t) \right] > 0. \end{aligned}$$

Then, by the monotonicity of λ , there exists a unique constant $l \in \left(0, \int_0^1 y(s) ds \right)$ such that $\lambda(l) = 0$, that is, (2.6) is satisfied for $\theta = \tilde{\varphi}_0 = l$, and there exists a constant $\rho \in (0, 1)$ such that $l = \int_0^\rho y(r) dr$. \square

Remark. For $\tilde{\varphi}_0 = l = \int_0^\rho y(r) dr$, we can rewrite the solution $x(t)$ of the BVP (2.2)–(2.3), given in (2.4) and (2.5) as

$$x(t) = \frac{a\Phi_q \left(\int_0^\rho y(r) dr \right) + \int_0^1 \int_0^t \Phi_q \left(\int_s^\rho y(r) dr \right) ds dA(t)}{1 - \alpha[1]} + \int_0^t \Phi_q \left(\int_s^\rho y(r) dr \right) ds$$

or

$$x(t) = \frac{b\Phi_q\left(\int_{\rho}^1 y(r)dr\right) + \int_0^1 \int_t^1 \Phi_q\left(\int_{\rho}^s y(r)dr\right) ds dB(t)}{1 - \beta[1]} + \int_t^1 \Phi_q\left(\int_{\rho}^s y(r)dr\right) ds$$

respectively.

In what follows, we consider the cone K in X defined by

$$K = \{x \in X : x(t) \geq 0 \text{ and } x(t) \text{ is concave on } [0, 1], \alpha[x] \geq 0, \text{ and } \beta[x] \geq 0\}.$$

Now if we define an operator $T : K \rightarrow K$ by

$$Tx(t) = \begin{cases} \frac{a\Phi_q\left(\int_0^{\rho} g(r)f(r, x(r), x'(r))dr\right) + \int_0^1 \int_0^s \Phi_q\left(\int_{\theta}^{\rho} g(r)f(r, x(r), x'(r))dr\right) d\theta dA(s)}{1 - \alpha[1]} \\ \quad + \int_0^t \Phi_q\left(\int_s^{\rho} g(r)f(r, x(r), x'(r))dr\right) ds, & 0 \leq s, t \leq \rho, \\ \frac{b\Phi_q\left(\int_{\rho}^1 g(r)f(r, x(r), x'(r))dr\right) + \int_0^1 \int_s^1 \Phi_q\left(\int_{\rho}^{\theta} g(r)f(r, x(r), x'(r))dr\right) d\theta dB(s)}{1 - \beta[1]} \\ \quad + \int_t^1 \Phi_q\left(\int_{\rho}^s g(r)f(r, x(r), x'(r))dr\right) ds, & \rho \leq s, t \leq 1, \end{cases} \quad (2.17)$$

then in view of Remark 2, a function $x(t)$ is a solution of BVP (1.1)–(1.2) if and only if $x(t)$ is a fixed point of the operator T given in (2.17).

Remark. From (2.17), we see that $Tx(t) \geq 0$ for $t \in [0, 1]$, and

$$(Tx)'(t) = \Phi_q\left(\int_t^{\rho} g(r)f(r, x(r), x'(r))dr\right) \geq 0 \text{ for } t \leq \rho$$

and

$$(Tx)'(t) = -\Phi_q\left(\int_{\rho}^t g(r)f(r, x(r), x'(r))dr\right) \leq 0 \text{ for } t \geq \rho.$$

Hence

$$\max_{0 \leq t \leq 1} Tx(t) = Tx(\rho).$$

For $x \in K$, we may proceed along the lines of the proof of Lemma 2.2 in [6] to obtain a Harnack type inequality as given in the following lemma.

Lemma 2.3. *Let $x(t) \geq 0$ be concave on $[0, 1]$. Then for any $\delta \in (0, 1/2)$, we have*

$$\min_{t \in [\delta, 1-\delta]} x(t) \geq \delta \max_{0 \leq t \leq 1} x(t) = \delta \|x\|_{\infty}.$$

Now, we list few properties of the classical fixed point index for compact maps [4]. Let K be a cone in a Banach space X . If Ω is a bounded open subset of K (in the relative topology), we denote by $\bar{\Omega}$ and $\partial\Omega$ the closure and boundary relative to K . When U is an open bounded subset of X , we write $U_K = U \cap K$ which is an open subset of K .

Theorem 2.4. *Let U be an bounded open set with $U_K \neq \phi$ and $\bar{U}_K \neq K$. Assume that $T : \bar{U}_K \rightarrow K$ is a compact map such that $x \neq Tx$ for $x \in \partial U_K$. Then the fixed point index $i_K(T, U_K)$ has the following properties:*

- (1) *If there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for all $x \in \partial U_K$ and all $\lambda > 0$, then $i_K(T, U_K) = 0$.*
- (2) *If $\|T\| \leq \|x\|$ for all $x \in \partial U_K$, then $i_K(T, U_K) = 1$*
- (3) *Let V be open in X with $\bar{V} \subset U_K$. If $i_K(T, U_K) = 1$ and $i_K(T, V_K) = 0$, then T has a fixed point in $U_K \setminus \bar{V}_K$. The same results holds if $i_K(T, U_K) = 0$ and $i_K(T, V_K) = 1$.*

We end this section with the following important lemma that we will use in the proof of our main result (Theorem 3.1 below).

Lemma 2.5. *Let conditions (A1)–(A5) hold. Then the mapping T defined in (2.17) is compact and continuous.*

Proof. The verification of the continuity of T is straight forward and hence we omit the details. Clearly, for $x \in K$, $Tx \geq 0$ for all $t \in [0, 1]$. Also, Tx is a concave function. Since

$$\begin{aligned} \alpha[Tx] &= \int_0^1 Tx(t) dA(t) \\ &= \frac{\alpha[1]}{(1 - \alpha[1])} \left[a\Phi_q \left(\int_0^\rho g(r)f(r, x(r), x'(r))dr \right) \right. \\ &\quad \left. + \int_0^1 \int_0^s \Phi_q \left(\int_\theta^\rho g(r)f(r, x(r), x'(r))dr \right) d\theta dA(s) \right] \\ &\quad + \frac{\alpha[1]}{(1 - \alpha[1])} \left[\int_0^1 \int_0^t \Phi_q \left(\int_s^\rho g(r)f(r, x(r), x'(r))dr \right) ds dA(t) \right] \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} \beta[Tx] &= \int_0^1 Tx(t) dB(t) \\ &= \frac{\beta[1]}{(1 - \alpha[1])} \left[a\Phi_q \left(\int_0^\rho g(r)f(r, x(r), x'(r))dr \right) \right. \\ &\quad \left. + \int_0^1 \int_0^s \Phi_q \left(\int_\theta^\rho g(r)f(r, x(r), x'(r))dr \right) d\theta dA(s) \right] \\ &\quad + \frac{\beta[1]}{(1 - \alpha[1])} \left[\int_0^1 \int_0^t \Phi_q \left(\int_s^\rho g(r)f(r, x(r), x'(r))dr \right) ds dB(t) \right] \\ &\geq 0 \end{aligned}$$

for $0 \leq s, t \leq \rho$, and

$$\begin{aligned} \alpha[Tx] &= \int_0^1 Tx(t) dA(t) \\ &= \frac{\alpha[1]}{(1 - \beta[1])} \left[b\Phi_q \left(\int_\rho^1 g(r)f(r, x(r), x'(r))dr \right) \right. \\ &\quad \left. + \int_0^1 \int_s^1 \Phi_q \left(\int_\rho^\theta g(r)f(r, x(r), x'(r))dr \right) d\theta dB(s) \right] \\ &\quad + \frac{\alpha[1]}{(1 - \alpha[1])} \left[\int_0^1 \int_t^1 \Phi_q \left(\int_\rho^s g(r)f(r, x(r), x'(r))dr \right) ds dA(t) \right] \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} \beta[Tx] &= \int_0^1 Tx(t) dB(t) \\ &= \frac{\beta[1]}{(1 - \beta[1])} \left[b\Phi_q \left(\int_\rho^1 g(r)f(r, x(r), x'(r))dr \right) \right. \\ &\quad \left. + \int_0^1 \int_s^1 \Phi_q \left(\int_\rho^\theta g(r)f(r, x(r), x'(r))dr \right) d\theta dB(s) \right] \\ &\quad + \frac{\beta[1]}{(1 - \alpha[1])} \left[\int_0^1 \int_t^1 \Phi_q \left(\int_\rho^s g(r)f(r, x(r), x'(r))dr \right) ds dB(t) \right] \\ &\geq 0 \end{aligned}$$

for $\rho \leq s, t \leq 1$ holds, we see that $T(K) \subset K$.

Next, we wish to prove that $T : K \rightarrow K$ is completely continuous. Let $\gamma > 0$ be any real number and set

$$\Omega_\gamma = \{x \in K : \|x\|_{C^1} < \gamma\};$$

then Ω_γ is an bounded open set in K . Hence, by (A4), for any $x \in \Omega_\gamma$ and $t \in [0, 1]$, we have

$$\begin{aligned} \|Tx\|_\infty &= Tx(\rho) \\ &= \frac{b\Phi_q \left(\int_\rho^1 g(r)f(r, x(r), x'(r))dr \right) + \int_0^1 \int_s^1 \Phi_q \left(\int_\rho^\theta g(r)f(r, x(r), x'(r))dr \right) d\theta dB(s)}{1 - \beta[1]} \\ &\quad + \int_\rho^1 \Phi_q \left(\int_\rho^s g(r)f(r, x(r), x'(r))dr \right) ds \end{aligned}$$

$$\begin{aligned} & b\Phi_q \left(\int_{\rho}^1 g(r)p_f(r)dr \right) + \int_0^1 \int_s^1 \Phi_q \left(\int_{\rho}^{\theta} g(r)p_f(r)dr \right) d\theta dB(s) \\ \leq & \frac{\phantom{b\Phi_q \left(\int_{\rho}^1 g(r)p_f(r)dr \right) + \int_0^1 \int_s^1 \Phi_q \left(\int_{\rho}^{\theta} g(r)p_f(r)dr \right) d\theta dB(s)}}{1 - \beta[1]} \\ & + \int_{\rho}^1 \Phi_q \left(\int_{\rho}^s g(r)p_f(r)dr \right) ds \end{aligned}$$

for $\rho \leq s$, and

$$\begin{aligned} \|Tx\|_{\infty} &= Tx(\rho) \\ &= \frac{\alpha\Phi_q \left(\int_0^{\rho} g(r)f(r, x(r), x'(r))dr \right) + \int_0^1 \int_0^s \Phi_q \left(\int_{\theta}^{\rho} g(r)f(r, x(r), x'(r))dr \right) d\theta dA(s)}{1 - \alpha[1]} \\ &+ \int_0^t \Phi_q \left(\int_s^{\rho} g(r)f(r, x(r), x'(r))dr \right) ds \\ &\leq \frac{\alpha\Phi_q \left(\int_0^{\rho} g(r)p_f(r)dr \right) + \int_0^1 \int_0^s \Phi_q \left(\int_{\theta}^{\rho} g(r)p_f(r)dr \right) d\theta dA(s)}{1 - \alpha[1]} \\ &+ \int_0^{\rho} \Phi_q \left(\int_s^{\rho} g(r)p_f(r)dr \right) ds \end{aligned}$$

for $s \leq \rho$. Also, from the facts that

$$(Tx)'(t) = \Phi_q \left(\int_t^{\rho} g(r)f(r, x(r), x'(r))dr \right) \geq 0 \text{ for } t \leq \rho$$

and

$$(Tx)'(t) = -\Phi_q \left(\int_{\rho}^t g(r)f(r, x(r), x'(r))dr \right) \leq 0 \text{ for } t \geq \rho,$$

we have

$$|(Tx)'(t)| \leq \Phi_q \left(\int_0^1 g(r)f(r, x(r), x'(r))dr \right).$$

Thus, for $\rho \leq s$ or $s \leq \rho$, we have from (A4), that $\|Tx\|_{C^1} < \infty$, which implies that T is uniformly bounded. Next, for every $\epsilon > 0$, there exists a $\delta \in \left(0, \frac{\epsilon}{\Phi_q(\int_0^1 g(s)p_f(s)ds)}\right)$ such that for any $t_1, t_2 \in [0, 1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned} |Tx(t_2) - Tx(t_1)| &\leq \left| \int_{t_1}^{t_2} \Phi_q \left(\int_0^1 g(r)f(r, x(r), x'(r))dr \right) ds \right| \\ &\leq |t_1 - t_2| \Phi_q \left(\int_0^1 g(r)f(r, x(r), x'(r))dr \right) \\ &< \delta \Phi_q \left(\int_0^1 g(r)p_f(r)dr \right) \end{aligned}$$

$< \epsilon$.

Hence, T is equicontinuous.

The equicontinuity of $\{(Tx)'(t)\}$ can be shown in a similar fashion. Therefore, T is relatively compact on $\overline{\Omega}_\gamma$, and hence compact on $\overline{\Omega}_\gamma$. Therefore, $T : K \rightarrow K$ is compact and continuous. \square

3. MAIN RESULTS

In this section, we shall apply Theorem 2.4 to prove our main results. For any $\delta \in (0, 1/2)$ denote the three real numbers L , M_1 , and M_2 by

$$L = \Phi_q \left(\int_0^1 g(r) dr \right), \quad M_1 = \min \left\{ \delta \Phi_q \left(\int_\delta^{1-\delta} g(r) dr \right), \int_0^\delta \Phi_q \left(\int_s^\delta g(r) dr \right) ds \right\}$$

and

$$M_2 = \int_{1-\delta}^1 \Phi_q \left(\int_{1-\delta}^s g(r) dr \right) ds.$$

Theorem 3.1. *Let $\delta \in (0, 1/2)$ be any real number and assume that (A1)–(A5). In addition, assume that*

(A6) *for any $x \in [0, \infty)$ and $y \in \mathbb{R}$, the mapping $t \mapsto f(t, x, y)$ is decreasing.*

(A7) *for any $t \in [0, 1]$, the mapping $x \mapsto f(t, x, y)$ is increasing.*

(A8) *there exists constant r_i , $i = 1, 2, 3$, with*

$$0 < r_1 < r_2 < \frac{r_2}{\delta} < r_3$$

such that

$$f(0, r_i, y) < \min \left\{ \Phi_p \left(\frac{r_i(1 - \beta[1])}{(1 + b)L} \right), \Phi_p \left(\frac{r_i}{L} \right) \right\} \text{ for } r_i \leq y \leq r_i, \quad i = 1, 3,$$

$$f(1 - \delta, x, y) > \Phi_p \left(\frac{r_2}{M_1} \right) \text{ for } r_2 \leq x \leq \frac{r_2}{\delta} \text{ and } \frac{r_2}{\delta} \leq y \leq \frac{r_2}{\delta},$$

and

$$f(1, x, y) > \Phi_p \left(\frac{r_2}{M_2} \right) \text{ for } r_2 \leq x \leq \frac{r_2}{\delta} \text{ and } \frac{r_2}{\delta} \leq y \leq \frac{r_2}{\delta}.$$

Then the BVP (1.1)–(1.2) has at least two positive solutions $x_1(t)$ and $x_2(t)$ with $0 < r_1 < \|x_1\| < r_2/\delta < \|x_2\| < r_3$.

Proof. We shall show that T satisfies all the conditions of Theorem 2.4. By Lemma 2.5, $T : K \rightarrow K$ is compact and continuous. Set

$$\Omega_{r_i} = \{x \in K : \|x\|_{C^1} < r_i\} \quad i = 1, 2, 3;$$

then for any $x \in \partial\Omega_{r_i}$, $i = 1, 2, 3$, we have $0 \leq x(t) \leq \|x\| = r_i$, $t \in [0, 1]$. Hence, $-r_i \leq x'(t) \leq r_i$, $i = 1, 3$.

First we consider the case $i = 1$ and 3. For $x \in \partial\Omega_{r_i}$, $i = 1$ and 3. we have

$$\begin{aligned} \|Tx\|_\infty &= \max_{0 \leq t \leq 1} Tx(t) = Tx(\rho) \\ &= \frac{b\Phi_q \left(\int_\rho^1 g(r) f(r, x(r), x'(r)) dr \right) + \int_0^1 \int_s^1 \Phi_q \left(\int_\rho^\theta g(r) f(r, x(r), x'(r)) dr \right) d\theta dB(s)}{1 - \beta[1]} \end{aligned}$$

$$\begin{aligned}
& + \int_{\rho}^1 \Phi_q \left(\int_{\rho}^s g(r) f(r, x(r), x'(r)) dr \right) ds \\
& \leq \frac{b \Phi_q \left(\int_0^1 g(r) f(r, x(r), x'(r)) dr \right) + \beta[1] \Phi_q \left(\int_0^1 g(r) f(r, x(r), x'(r)) dr \right)}{(1 - \beta[1])} \\
& \quad + \Phi_q \left(\int_0^1 g(r) f(r, x(r), x'(r)) dr \right) \\
& \leq \frac{(1+b)}{(1 - \beta[1])} \Phi_q \left(\int_0^1 g(r) f(r, x(r), x'(r)) dr \right) \\
& \leq \frac{(1+b)}{(1 - \beta[1])} \Phi_q \left(\int_0^1 g(r) f(0, x(r), x'(r)) dr \right) \\
& \leq \frac{(1+b)}{(1 - \beta[1])} \Phi_q \left(\int_0^1 g(r) f(0, r_i, x'(r)) dr \right) \\
& < r_i
\end{aligned}$$

and

$$\begin{aligned}
|(Tx)'| & \leq \Phi_q \left(\int_0^1 g(r) f(0, x(r), x'(r)) dr \right) \\
& \leq \Phi_q \left(\int_0^1 g(r) f(0, r_i, x'(r)) dr \right) \\
& \leq r_i,
\end{aligned}$$

which implies that $\|Tx\|_{C^1} < r_i = \|x\|_{C^1}$ for $i = 1, 3$. This indicates that $\|Tx\|_{C^1} < \|x\|_{C^1}$ for any $x \in \partial\Omega_{r_i}$ with $i = 1, 3$. By Theorem 2.4 (2), we have $i_K(T, \Omega_{r_i}) = 1$, $i = 1, 3$.

Next, for the case $i = 2$, we consider the set

$$V_{r_2} = \left\{ x \in K : \min_{t \in [\delta, 1-\delta]} x(t) < r_2 \right\}.$$

Then $\Omega_{r_2} \subset V_{r_2} \subset \Omega_{r_2/\delta}$ and $\min_{t \in [\delta, 1-\delta]} x(t) = r_2$ for $x \in \partial V_{r_2}$. Further, by Lemma 2.3, we have for any $x \in K$,

$$\max_{0 \leq t \leq 1} x(t) \leq \frac{1}{\delta} \min_{t \in [\delta, 1-\delta]} x(t) = \frac{r_2}{\delta}.$$

So for any $x \in \partial V_{r_2}$, we obtain

$$r_2 = \min_{t \in [\delta, 1-\delta]} x(t) \leq x(t) \leq \max_{0 \leq t \leq 1} x(t) \leq \frac{r_2}{\delta}. \quad (3.1)$$

Thus, $-\frac{r_2}{\delta} \leq x' \leq \frac{r_2}{\delta}$ for $\delta \leq t \leq 1 - \delta$. Let $e(t) \equiv 1$ for $t \in [0, 1]$. Then, it is clear that $e \in K \setminus \{0\}$. We claim that $x \neq Tx + \lambda e$ for all $x \in \partial V_{r_2}$ and all $\lambda \geq 0$. To the contrary,

suppose that there exists $x^*(t) \in \partial V_{r_2}$ and $\lambda^* \geq 0$ such that

$$x^*(t) = Tx^*(t) + \lambda^* e(t).$$

Then,

$$r_2 = \min_{t \in [\delta, 1-\delta]} x^*(t) = \min_{t \in [\delta, 1-\delta]} Tx^*(t) + \lambda^*. \quad (3.2)$$

We consider three cases depending on the location of ρ in $[0, 1]$, and obtain contradictions in each case. First suppose that $\rho \in [\delta, 1-\delta]$. Then we have, either $\min_{t \in [\delta, 1-\delta]} Tx^*(t) = x^*(\delta)$ or $\min_{t \in [\delta, 1-\delta]} Tx^*(t) = x^*(1-\delta)$. If $\min_{t \in [\delta, 1-\delta]} Tx^*(t) = x^*(\delta)$, then from (3.2) we have

$$\begin{aligned} r_2 &= \min_{t \in [\delta, 1-\delta]} Tx^*(t) + \lambda^* \\ &\geq Tx^*(\delta) + \lambda^* \\ &= \frac{\alpha \Phi_q \left(\int_0^\rho g(r) f(r, x^*(r), x^{*'}(r)) dr \right)}{1 - \alpha[1]} \\ &\quad + \frac{\int_0^1 \int_0^s \Phi_q \left(\int_\theta^\rho g(r) f(r, x^*(r), x^{*'}(r)) dr \right) ds dA(s)}{1 - \alpha[1]} \\ &\quad + \int_0^\delta \Phi_q \left(\int_s^\rho g(r) f(r, x^*(r), x^{*'}(r)) dr \right) ds + \lambda^* \\ &\geq \int_0^\delta \Phi_q \left(\int_s^\delta g(r) f(r, x^*(r), x^{*'}(r)) dr \right) ds + \lambda^* \\ &\geq \int_0^\delta \Phi_q \left(\int_s^\delta g(r) f(1-\delta, x^*(r), x^{*'}(r)) dr \right) ds + \lambda^*. \end{aligned}$$

Hence, for $x^* \in \partial V_{r_2}$, using (3.1) and (A8), we obtain the contraction that $r_2 > r_2 + \lambda^*$. If $\min_{t \in [\delta, 1-\delta]} Tx^*(t) = x^*(1-\delta)$, then from (3.2) we have

$$\begin{aligned} r_2 &= \min_{t \in [\delta, 1-\delta]} Tx^*(t) + \lambda^* \\ &\geq Tx^*(1-\delta) + \lambda^* \\ &= \frac{b \Phi_q \left(\int_\rho^1 g(r) f(r, x^*(r), x^{*'}(r)) dr \right)}{1 - \beta[1]} \\ &\quad + \frac{\int_0^1 \int_s^1 \Phi_q \left(\int_\rho^\theta g(r) f(r, x^*(r), x^{*'}(r)) dr \right) d\theta dB(s)}{1 - \beta[1]} \\ &\quad + \int_{1-\delta}^1 \Phi_q \left(\int_\rho^s g(r) f(r, x^*(r), x^{*'}(r)) dr \right) ds + \lambda^* \end{aligned}$$

$$\begin{aligned}
&\geq \int_{1-\delta}^1 \Phi_q \left(\int_{1-\delta}^s g(r) f(r, x^*(r), x^{*'}(r)) dr \right) ds + \lambda^* \\
&\geq \int_{1-\delta}^1 \Phi_q \left(\int_{1-\delta}^s g(r) f(1, x^*(r), x^{*'}(r)) dr \right) ds + \lambda^* \\
&> r_2 + \lambda^*,
\end{aligned}$$

which is a contradiction.

Next, suppose that $\rho \in [1 - \delta, 1]$. Then from (3.2), we have

$$\begin{aligned}
r_2 &= \min_{t \in [\delta, 1-\delta]} Tx^*(t) + \lambda^* \\
&= Tx^*(\delta) + \lambda^* \\
&\geq \int_0^\delta \Phi_q \left(\int_s^\rho g(r) f(r, x^*(r), x^{*'}(r)) dr \right) ds + \lambda^* \\
&\geq \int_0^\delta \Phi_q \left(\int_s^{1-\delta} g(r) f(1 - \delta, x^*(r), x^{*'}(r)) dr \right) ds + \lambda^* \\
&\geq \int_0^\delta \Phi_q \left(\int_{\delta}^{1-\delta} g(r) f(1 - \delta, x^*(r), x^{*'}(r)) dr \right) ds + \lambda^*, \quad (\because s \leq \delta) \\
&> r_2 + \lambda^*,
\end{aligned}$$

which again is a contradiction.

Finally, suppose that $\rho \in [0, \delta)$. From (3.2), we have

$$\begin{aligned}
r_2 &= \min_{t \in [\delta, 1-\delta]} Tx^*(t) + \lambda^* e \\
&= Tx^*(1 - \delta) + \lambda^* \\
&\geq \int_{1-\delta}^1 \Phi_q \left(\int_\rho^s g(r) f(r, x^*(r), x^{*'}(r)) dr \right) ds + \lambda^* \\
&\geq \int_{1-\delta}^1 \Phi_q \left(\int_{1-\delta}^s g(r) f(r, x^*(r), x^{*'}(r)) dr \right) ds + \lambda^* \\
&\geq \int_{1-\delta}^1 \Phi_q \left(\int_{1-\delta}^s g(r) f(1, x^*(r), x^{*'}(r)) dr \right) ds + \lambda^* \\
&> r_2 + \lambda^*,
\end{aligned}$$

a contradiction.

Hence, our claim holds, that is, $x \neq Tx + \lambda e$ for all $x \in \partial V_{r_2}$ and all $\lambda \geq 0$. By Theorem 2.4(a), we see that $i_K(T, V_{r_2}) = 0$, so by Theorem 2.4(c), the operator T has two fixed points x_1 and x_2 that in turn are positive solutions of the BVP (1.1)–(1.2) satisfying $x_1, x_2 \in K$ with $r_1 < \|x_1\|_\infty < r_2/\delta < \|x_2\|_\infty < r_3$. This completes the proof of the theorem. \square

By proceeding as in the lines of the proof of Theorem 3.1, we can prove the following theorem; we omit the details.

Theorem 3.2. *In addition to conditions (A1)–(A7) assume that*

(A9): *there exists constant r_i , $i = 1, 2, 3$ with*

$$0 < r_1 < \frac{r_1}{\delta} < r_2 < \frac{r_2}{\delta} < r_3$$

such that

$$f(0, r_2, y) < \min \left\{ \Phi_p \left(\frac{r_2(1 - \beta[1])}{(1+b)L} \right), \Phi_p \left(\frac{r_2}{L} \right) \right\} \text{ for } -r_2/\delta \leq y \leq r_2/\delta,$$

$$f(1 - \delta, r_i, y) > \Phi_p \left(\frac{r_i}{M_1} \right) \text{ for } -r_i/\delta \leq y \leq r_i/\delta, i = 1, 3,$$

and

$$f(1, r_i, y) > \Phi_p \left(\frac{r_i}{M_2} \right) \text{ for } -r_i/\delta \leq y \leq r_i/\delta, i = 1, 3$$

are satisfied. Then the BVP (1.1)–(1.2) has at least two positive solution x_1, x_2 with $\frac{r_1}{\delta} < \|x_1\|_\infty < r_2$ and $\frac{r_2}{\delta} < \|x_2\|_\infty < \frac{r_3}{\delta}$.

4. DISCUSSION AND EXAMPLES

As mentioned earlier, the Riemann-Stieltjes integrals $\alpha[x]$ and $\beta[x]$ are quite general and include a variety of nonlocal boundary conditions. For example:

- (i) If $\alpha[x] = \alpha x(\eta)$, $0 < \eta < 1$ and $\beta[x] = \beta x(\mu)$, $0 < \mu < 1$, then condition (A2) reduces to $0 < \alpha < 1$ and $0 < \beta < 1$,
- (ii) If $\alpha[x] = \frac{\alpha}{\eta_2 - \eta_1} \int_{\eta_1}^{\eta_2} \alpha t x(t) dt$, $0 < \eta_1 < \eta_2 < 1$ and $\beta[x] = \frac{\beta}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \alpha t x(t) dt$, $0 < \mu_1 < \mu_2 < 1$, α and β are positive constants, then (A2) becomes $0 < \alpha(\eta_1 + \eta_2) < 2$ and $0 < \beta(\mu_1 + \mu_2) < 2$.
- (iii) If $\alpha[x] = \alpha \int_0^1 t^m x(t) dt$ and $\beta[x] = \beta \int_0^1 t^n x(t) dt$, $m, n > -1$, then (A2) reduces to $0 < \alpha < m + 1$ and $0 < \beta < n + 1$.

Example 4.1. Let $p > 1$ and $q > 1$ be such that $1/p + 1/q = 1$. Let $\delta \in (0, 1/2)$ and assume that (A2) is satisfied. Set

$$\phi_1 := \Phi_p \left(\frac{r_1(1 - \beta[1])}{(1+b)L} \right), \quad \phi_2 := \Phi_p \left(\frac{r_2}{M_1} \right),$$

$$\tilde{\phi}_2 := \Phi_p \left(\frac{r_2}{M_2} \right), \quad \text{and} \quad \phi_3 := \Phi_p \left(\frac{r_3(1 - \beta[1])}{(1+b)L} \right),$$

where r_1, r_2 , and r_3 are chosen such that $0 < r_1 < r_2 < r_2/\delta < r_3$ and the inequalities

$$\begin{cases} r_1^2 < \frac{7225(3-\delta)}{2(1+\delta)} \phi_1, \\ r_3^2 < \frac{7225((5-3\delta)\phi_3 - (1+\delta)\phi_2)}{4(1+\delta)}, \\ \phi_3 > \left(\frac{4-3\delta}{\delta} \right) \max\{\phi_2, \tilde{\phi}_2\} \end{cases} \quad (4.1)$$

are satisfied. Set

$$f_1(x, y) = \frac{\phi_1}{2} \left(\frac{1}{2} - \frac{1}{2} \cos \left(\frac{\pi x}{r_1} \right) \right) + \frac{y^2}{7225},$$

$$f_2(x, y) = \frac{1}{8} (2\phi_1 + 3\phi_2 + \phi_3) + \frac{1}{8} (2\phi_1 - 3\phi_2 - \phi_3) \cos \left(\frac{\pi(x - r_1)}{(r_2 - r_1)} \right) + \frac{y^2}{7225},$$

$$f_3(x, y) = \frac{1}{8}(5\phi_2 + 3\phi_3) + \frac{1}{8}(\phi_2 - \phi_3) \cos\left(\frac{\pi(x - r_2)}{(r_2/\delta - r_2)}\right) + \frac{y^2}{7225},$$

$$f_4(x, y) = \frac{1}{8}(3\phi_2 + 5\phi_3) + \frac{1}{8}(\phi_2 - \phi_3) \cos\left(\frac{\pi(x - r_2/\delta)}{(r_3 - r_2/\delta)}\right) + \frac{y^2}{7225},$$

and for $0 \leq t < 1$, let

$$f(t, x, y) = \frac{(1 + \delta - t)}{2} \begin{cases} f_1(x, y), & 0 \leq x \leq r_1, \\ f_2(x, y), & r_1 \leq x \leq r_2, \\ f_3(x, y), & r_2 \leq x \leq r_2/\delta, \\ f_4(x, y), & r_2/\delta \leq x \leq r_3. \end{cases} \quad (4.2)$$

It is easy to see that conditions (A6) and (A7) are satisfied. Using (4.1), we have

$$f(0, r_1, y) = \frac{(1 + \delta)}{2} \left[\frac{\phi_1}{2} + \frac{r_1^2}{7225} \right] < \phi_1 := \Phi_p \left(\frac{r_1(1 - \beta[1])}{(1 + b)L} \right),$$

$$f(0, r_3, y) = \frac{(1 + \delta)}{2} \left[\frac{(\phi_2 + 3\phi_3)}{4} + \frac{r_3^2}{7225} \right] < \phi_3,$$

$$f(1 - \delta, x) \geq f(1 - \delta, r_2) = \delta \frac{(3\phi_2 + \phi_3)}{4} > \phi_2$$

and

$$f(1, x) \geq f(1, r_2) = \delta \frac{(3\phi_2 + \phi_3)}{4} > \tilde{\phi}_2$$

for $r_2 \leq x \leq r_2/\delta$, so Theorem 3.1 can be applied to the BVP (1.1)–(1.2), with f given in (4.2).

Next, we give a particular example to illustrate Theorem 3.1.

Example 4.2. Let $p = 3$, $\delta = 1/4$, $g(t) \equiv 1$, $b = 1$, and $\beta[x] = \frac{1}{2}x(\eta)$, $0 < \eta < 1$. Then $\beta[1] = \frac{1}{2}$, $q = \frac{3}{2}$, $L = 1$, $M_1 = \min\left\{\frac{1}{\sqrt{2}}, \frac{1}{12}\right\} = \frac{1}{12}$, and $M_2 = \frac{1}{12}$. Here $\frac{(1 - \beta[1])}{(1 + b)} = \frac{1}{4} < 1$. For $0 \leq t < 1$, let

$$f(t, x, y) = \frac{(1 + \delta - t)}{2} \begin{cases} \frac{1}{1024} - \frac{1}{1024} \cos 4\pi x + \frac{y^2}{7225}, & 0 \leq x \leq 1/4, \\ \frac{1212417}{1024} - \frac{1212415}{1024} \cos \pi(4x - 1) + \frac{y^2}{7225}, & 1/4 \leq x \leq 1/2, \\ 3264 - 896 \cos \frac{\pi}{3}(2x - 1) + \frac{y^2}{7225}, & 1/2 \leq x \leq 2, \\ 5056 - 896 \cos \frac{\pi}{350}(x - 2) + \frac{y^2}{7225}, & 2 \leq x \leq 352. \end{cases} \quad (4.3)$$

The conditions (A6) and (A7) are satisfied. Set $r_1 = 1/4$, $r_2 = 1/2$ and $r_3 = 352$; then $0 < r_1 < r_2 < r_2/\delta < r_3$,

$$\phi_1 := \Phi_p \left(\frac{r_1(1 - \beta[1])}{(1 + b)L} \right) = \frac{1}{256}, \quad \phi_2 := \Phi_p \left(\frac{r_2}{M_1} \right) = 576,$$

$$\tilde{\phi}_2 := \Phi_p \left(\frac{r_2}{M_2} \right) = 36 \quad \phi_3 := \Phi_p \left(\frac{r_3(1 - \beta[1])}{(1 + b)L} \right) = 7744$$

and the conditions in (4.1) are satisfied.

Since

$$\begin{aligned} f(0, r_1, y) &= f(0, 1/4, y) \leq \frac{5}{8} \left[\frac{1}{512} + \frac{1}{16 \times 7225} \right] \\ &< \frac{1}{256} = \Phi_p \left(\frac{r_1(1 - \beta[1])}{(1 + b)L} \right) \text{ for } -1/4 \leq y \leq 1/4, \\ f(0, r_3, y) &= f(0, 352, y) \leq \frac{5}{8} \left[5952 + \frac{123904}{7225} \right] = 3318.218 \\ &< 7744 = \Phi_p \left(\frac{r_3(1 - \beta[1])}{(1 + b)L} \right) \text{ for } -352 \leq y \leq 532, \end{aligned}$$

$$f(1 - \delta, x, y) \geq f(1 - \delta, r_2, y) = f(3/4, 1/2, y) > 592 > 36 = \Phi_p \left(\frac{r_2}{M_1} \right)$$

for $-1/8 \leq y \leq 1/8$, and

$$f(1, x, y) \geq f(1, r_2, y) = f(1, 1/2, y) > 296 > 36 = \Phi_p \left(\frac{r_2}{M_2} \right)$$

for $-1/8 \leq y \leq 1/8$, Theorem 3.1 can be applied to the BVP

$$\begin{cases} (\Phi_p(x'))' + f(t, x, x') = 0, & 0 \leq t \leq 1, \\ x(0) - \alpha x'(0) = \alpha x(\eta), & 0 < \eta < 1, \\ x(1) + x'(1) = \frac{1}{2}x(\mu), & 0 < \mu < 1, \end{cases} \quad (4.4)$$

with f given in (4.3). By Theorem 3.1, the problem (4.4) has at least two positive solutions x_1, x_2 with $1/4 \leq \|x_1\| \leq 2$ and $2 \leq \|x_2\| \leq 352$.

Remark. The piecewise continuous function f given in (4.3) can also be replaced by

$$\begin{aligned} f(t, x, y) &= \frac{(1 + \delta - t)}{2} \left[\frac{128203375523}{3808115850} x^4 - \frac{2753271348983}{230794900} x^3 \right. \\ &\quad \left. + \frac{106156471625653}{3808115850} x^2 - \frac{3950296709102}{634685975} x + \frac{y^2}{7225} \right]. \end{aligned} \quad (4.5)$$

We can then apply Theorem 3.1 to the problem (4.4) with f given in (4.5) to show that the problem has at least two positive solutions x_1, x_2 with $1/4 \leq \|x_1\| \leq 2$ and $2 \leq \|x_2\| \leq 352$.

In Example 4.1, a general result was obtained for the existence of positive solutions of (1.1)–(1.2), with f given in (4.2), $p, q > 1$, $0 < r_1 < r_2 < r_2/\delta < r_3$, and the constants $\phi_1, \phi_2, \tilde{\phi}_2$, and ϕ_3 satisfying the inequalities in (4.1). We conclude this paper with example for the case $p = q = 2$.

Example 4.3. Consider the boundary value problem

$$\begin{cases} x'' + f(t, x, x') = 0, & 0 \leq t \leq 1, \\ x(0) - \alpha x'(0) = \alpha[x], \\ x(1) + x'(1) = \beta[x]. \end{cases} \quad (4.6)$$

Let $g(t) \equiv 1$, $b = 1$, and $\beta[x] = \frac{1}{2}x(\eta)$, $0 < \eta < 1$. Then, $\beta[1] = \frac{1}{2}$, $q = 2$, $L = 1$, $M_1 = \frac{1}{32}$, and $M_2 = \frac{1}{32}$. If we set $r_1 = 1/4$, $r_2 = 1/2$, $r_3 = 1860$, and

$$f(t, x, y) = \frac{(1 + \delta - t)}{2} \begin{cases} \frac{1}{64} - \frac{1}{64} \cos 4\pi x + \frac{y^2}{8850}, & 0 \leq x \leq 1/4, \\ \frac{4105}{64} - \frac{4103}{64} \cos \pi(4x - 1) + \frac{y^2}{8850}, & 1/4 \leq x \leq 1/2, \\ \frac{1475}{8} - \frac{449}{8} \cos \frac{\pi}{3}(2x - 1) + \frac{y^2}{8850}, & 1/2 \leq x \leq 2, \\ \frac{2373}{8} - \frac{449}{8} \cos \frac{\pi}{1858}(x - 2) + \frac{y^2}{8850}, & 2 \leq x \leq 1860, \end{cases} \quad (4.7)$$

then the inequalities

$$\begin{aligned} f(0, 1/4, y) &< \frac{1}{16} \quad \text{for } -1/4 \leq y \leq 1/4, \\ f(0, 1860, y) &< 464.790784 < 465 \quad \text{for } -1860 \leq y \leq 1860, \\ f(3/4, 1/2, y) &\geq \frac{513}{16} > 16 \quad \text{for } -1/8 \leq y \leq 1/8, \end{aligned}$$

and

$$f(1, 1/2, y) \geq \frac{513}{32} \quad \text{for } -1/8 \leq y \leq 1/8$$

imply that all the conditions of Theorem 3.1 are satisfied, so the BVP (4.6) with f given in (4.7) has at least two positive solutions x_1, x_2 with $1/4 \leq \|x_1\| \leq 2$ and $2 \leq \|x_2\| \leq 1860$. We can also replace the function f in (4.7) by

$$\begin{aligned} f(t, x, y) = \frac{(1 + \delta - t)}{2} &\left[\frac{114789441720137}{334631973966780} x^4 - \frac{855116002574918627}{1338527895867120} x^3 \right. \\ &\left. + \frac{4027592955239820581}{2677055791734240} x^2 - \frac{449844227597169277}{1338527895867120} x + \frac{y^2}{8850} \right] \quad (4.8) \end{aligned}$$

and Theorem 3.1 can be applied to problem (4.6) with f is given in (4.8).

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SESHADEV PADHI

DEPARTMENT OF MATHEMATICS, BIRLA INSTITUTE OF TECHNOLOGY, MESRA, RANCHI - 835215, INDIA
Email address: spadhi@bitmesra.ac.in

JAFFAR ALI

DEPARTMENT OF MATHEMATICS, FLORIDA GULF COAST UNIVERSITY, FORTMYRES, FLORIDA 33965, USA
Email address: jashaulhameed@fgcu.edu

JOHN R. GRAEF

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE AT CHATTANOOGA, CHATTANOOGA, TN 37403, USA
Email address: john-graef@utc.edu