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HYBRID STRUCTURES AND APPLICATIONS

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ABSTRACT. The notion of hybrid structures is introduced, and several properties are investigated. Applications in BCK/BCI-algebras and linear spaces are discussed. Hybrid subalgebra, hybrid field and hybrid linear space are introduced, and related properties are investigated.

1. INTRODUCTION

The concept of fuzzy set is first introduced by Zadeh [21] in 1965. As a generalization of fuzzy sets, Torra introduced the concept of hesitant fuzzy sets ([19, 20]). The hesitant fuzzy set is very useful to express peoples hesitancy in daily life, and it is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Various problems in many fields involve data containing uncertainties which are dealt with wide range of existing theories such as the theory of probability, (intuitionistic) fuzzy set theory, vague sets, theory of interval mathematics and rough set theory etc. All of these theories have their own difficulties which are pointed out in [15]. To overcome these difficulties, Molodtsov [15] introduced the soft set theory as a new mathematical tool for dealing with uncertainties that is free from the difficulties. Molodtsov successfully applied the soft set theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement and so on (see [15, 16, 17, 18]). Soft set theory is applied to algebraic structures such as BCK/BCI-algebra ([8, 11, 12]), d-algebras ([10]), ring ([1]), semiring ([3, 6]), group (see [2]), ordered semigroup ([9]), decision making ([4, 5, 13]) and BL-algebra ([22]).

As a parallel circuit of fuzzy sets and soft sets (or, hesitant fuzzy sets), we introduced the notion of hybrid structure in a set of parameters over an initial universe set, and investigate several properties. Using this notion, we introduce the concepts of a hybrid subalgebra, a hybrid field and a hybrid linear space. We consider the hybrid union and hybrid intersection of hybrid subalgebras in BCK/BCI-algebras. We discuss characterizations of a hybrid subalgebra and a hybrid linear space. Given a hybrid subalgebra, we make a new hybrid subalgebra in BCK/BCI-algebras. We consider the hybrid image and preimage of a

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hybrid subalgebra and a hybrid linear space under the BCK/BCI-homomorphisms and the linear transformation of linear spaces, respectively.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $\mathcal{X} := (X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra $\mathcal{X}$ satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then $\mathcal{X}$ is called a *BCK-algebra*.

Any BCK/BCI-algebra $\mathcal{X}$ satisfies the following conditions:

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$. Note that $(\mathcal{X}, \leq)$ is a partially ordered set (see [14]).

A nonempty subset S of a BCK/BCI-algebra $\mathcal{X}$ is called a *subalgebra* of $\mathcal{X}$ if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [7, 14] for further information regarding BCK/BCI-algebras.

3. Hybrid structures

In this paper, we shall use convensions I for the unit interval, L for the set of parameters and $\mathscr{P}(U)$ for the power set of an initial universe set U.

Definition 3.1. A *hybrid structure* in L over U is defined to be a mapping

$$\tilde{f}_\lambda := (\tilde{f}, \lambda) : L \to \mathscr{P}(U) \times I,\ x \mapsto (\tilde{f}(x), \lambda(x))$$

where $\tilde{f} : L \to \mathscr{P}(U)$ and $\lambda : L \to I$ are mappings.

Example 3.2. Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be the set of six wooden houses under consideration and let L be the set of parameters which consist of "cheap", "expensive", "beautiful" and "in good location". Consider a hybrid structure in L over U which is given in Table 1.

Table 1. Tabular representation of the hybrid structure set $\tilde{f}_\lambda$

L	$\tilde{f}$	λ
cheap	$\{h_1, h_3\}$	0.3
expensive	$\{h_2, h_5\}$	0.7
beautiful	$\{h_2, h_3, h_5\}$	0.6
in good location	$\{h_2, h_4\}$	0.5

Then $\tilde{f}_\lambda$ is a hybrid structure in L over U.

Let us denote by $\mathbb{H}(L)$ the set of all hybrid structures in L over U. We define an order $\ll$ in $\mathbb{H}(L)$ as follows:

$$\Big(\forall \tilde{f}_\lambda, \tilde{g}_\gamma \in \mathbb{H}(L)\Big) \Big(\tilde{f}_\lambda \ll \tilde{g}_\gamma \ \Leftrightarrow\ \tilde{f} \,\tilde{\subseteq}\, \tilde{g},\ \lambda \succeq \gamma\Big) \tag{3.1}$$

where $\tilde{f} \,\tilde{\subseteq}\, \tilde{g}$ means that $\tilde{f}(x) \subseteq \tilde{g}(x)$ and $\lambda \succeq \gamma$ means that $\lambda(x) \geq \gamma(x)$ for all $x \in L$. Note that $(\mathbb{H}(L), \ll)$ is a poset.

Definition 3.3. Let $\tilde{f}_\lambda$ be a hybrid structure in L over U. Then the sets

$$\begin{aligned}
\tilde{f}_\lambda[\alpha, t] &:= \Big\{x \in L \mid \tilde{f}(x) \supseteq \alpha,\ \lambda(x) \leq t\Big\},\\
\tilde{f}_\lambda(\alpha, t] &:= \Big\{x \in L \mid \tilde{f}(x) \supsetneqq \alpha,\ \lambda(x) \leq t\Big\},\\
\tilde{f}_\lambda[\alpha, t) &:= \Big\{x \in L \mid \tilde{f}(x) \supseteq \alpha,\ \lambda(x) < t\Big\},\\
\tilde{f}_\lambda(\alpha, t) &:= \Big\{x \in L \mid \tilde{f}(x) \supsetneqq \alpha,\ \lambda(x) < t\Big\}
\end{aligned}$$

are called the $[\alpha, t]$-*hybrid cut*, $(\alpha, t]$-*hybrid cut*, $[\alpha, t)$-*hybrid cut*, and (α, t)-*hybrid cut* of $\tilde{f}_\lambda$, respectively, where $\alpha \in \mathscr{P}(U)$ and $t \in I$. Obviously,

$$\tilde{f}_\lambda(\alpha, t) \subseteq \tilde{f}_\lambda(\alpha, t] \subseteq \tilde{f}_\lambda[\alpha, t] \text{ and } \tilde{f}_\lambda(\alpha, t) \subseteq \tilde{f}_\lambda[\alpha, t) \subseteq \tilde{f}_\lambda[\alpha, t].$$

Definition 3.4. Let $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ be hybrid structures in L over U. Then the *hybrid intersection* of $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ is denoted by $\tilde{f}_\lambda \Cap \tilde{g}_\gamma$ and is defined to be a hybrid structure

$$\tilde{f}_\lambda \Cap \tilde{g}_\gamma : L \to \mathscr{P}(U) \times I,\ x \mapsto \Big((\tilde{f} \,\tilde{\cap}\, \tilde{g})(x), (\lambda \vee \gamma)(x)\Big)$$

for all $x \in L$, where

$$\begin{aligned}
&\tilde{f} \,\tilde{\cap}\, \tilde{g} : L \to \mathscr{P}(U),\ x \mapsto \tilde{f}(x) \cap \tilde{g}(x),\\
&\lambda \vee \gamma : L \to I,\ x \mapsto \bigvee \{\lambda(x), \gamma(x)\}.
\end{aligned} \tag{3.2}$$

Definition 3.5. Let $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ be hybrid structures in L over U. Then the *hybrid union* of $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ is denoted by $\tilde{f}_\lambda \Cup \tilde{g}_\gamma$ and is defined to be a hybrid structure

$$\tilde{f}_\lambda \Cup \tilde{g}_\gamma : L \to \mathscr{P}(U) \times I,\ x \mapsto \Big((\tilde{f} \,\tilde{\cup}\, \tilde{g})(x), (\lambda \wedge \gamma)(x)\Big),$$

where

$$\begin{aligned}
&\tilde{f} \,\tilde{\cup}\, \tilde{g} : L \to \mathscr{P}(U),\ x \mapsto \tilde{f}(x) \cup \tilde{g}(x),\\
&\lambda \wedge \gamma : L \to I,\ x \mapsto \bigwedge \{\lambda(x), \gamma(x)\}.
\end{aligned} \tag{3.3}$$

Proposition 3.1. *Let $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ be two hybrid structures in L over U. For any $\alpha, \beta \in \mathscr{P}(U)$ and $s, t \in I$, we have the following properties:*

(1) *If $\alpha \subseteq \beta$ and $t \leq s$, then $\tilde{f}_\lambda[\beta, t] \subseteq \tilde{f}_\lambda[\alpha, s]$.*
(2) *If $\alpha = \emptyset$ and $t = 1$, then $\tilde{f}_\lambda[\alpha, t] = L$.*
(3) *If $\tilde{f}_\lambda \ll \tilde{g}_\gamma$, then $\tilde{f}_\lambda[\alpha, t] \subseteq \tilde{g}_\gamma[\alpha, t]$.*
(4) $\Big(\tilde{f}_\lambda \Cap \tilde{g}_\gamma\Big)[\alpha, t] = \tilde{f}_\lambda[\alpha, t] \cap \tilde{g}_\gamma[\alpha, t]$.
(5) $\Big(\tilde{f}_\lambda \Cup \tilde{g}_\gamma\Big)[\alpha, t] \supseteq \tilde{f}_\lambda[\alpha, t] \cup \tilde{g}_\gamma[\alpha, t]$.

Proof. (1) Assume that $\alpha \subseteq \beta$ and $t \leq s$, and let $x \in \tilde{f}_\lambda[\beta, t]$. Then $\tilde{f}(x) \supseteq \beta \supseteq \alpha$ and $\lambda(x) \leq t \leq s$, which shows that $x \in \tilde{f}_\lambda[\alpha, s]$. Thus $\tilde{f}_\lambda[\beta, t] \subseteq \tilde{f}_\lambda[\alpha, s]$.

(2) It is clear.

(3) Assume that $\tilde{f}_\lambda \ll \tilde{g}_\gamma$ and let $x \in \tilde{f}_\lambda[\alpha, t]$. Then

$$\tilde{g}(x) \supseteq \tilde{f}(x) \supseteq \alpha \text{ and } \gamma(x) \leq \lambda(x) \leq t.$$

Hence $x \in \tilde{g}_\gamma[\alpha, t]$, and so $\tilde{f}_\lambda[\alpha, t] \subseteq \tilde{g}_\gamma[\alpha, t]$.

(4) For any $x \in L$, we have

$$\begin{aligned} x \in \left(\tilde{f}_\lambda \Cap \tilde{g}_\gamma\right)[\alpha, t] &\Leftrightarrow \left(\tilde{f} \tilde{\cap} \tilde{g}\right)(x) \supseteq \alpha,\ (\lambda \vee \gamma)(x) \leq t \\ &\Leftrightarrow \tilde{f}(x) \cap \tilde{g}(x) \supseteq \alpha,\ \bigvee\{\lambda(x), \gamma(x)\} \leq t \\ &\Leftrightarrow \tilde{f}(x) \supseteq \alpha, \tilde{g}(x) \supseteq \alpha, \lambda(x) \leq t, \gamma(x) \leq t \\ &\Leftrightarrow x \in \tilde{f}_\lambda[\alpha, t],\ x \in \tilde{g}_\gamma[\alpha, t] \\ &\Leftrightarrow x \in \tilde{f}_\lambda[\alpha, t] \cap \tilde{g}_\gamma[\alpha, t]. \end{aligned}$$

(5) Since $\tilde{f}_\lambda \ll \tilde{f}_\lambda \Cup \tilde{g}_\gamma$ and $\tilde{g}_\gamma \ll \tilde{f}_\lambda \Cup \tilde{g}_\gamma$, it follows from (3) that

$$\tilde{f}_\lambda[\alpha, t] \subseteq \left(\tilde{f}_\lambda \Cup \tilde{g}_\gamma\right)[\alpha, t] \text{ and } \tilde{g}_\gamma[\alpha, t] \subseteq \left(\tilde{f}_\lambda \Cup \tilde{g}_\gamma\right)[\alpha, t].$$

Therefore $\tilde{f}_\lambda[\alpha, t] \cup \tilde{g}_\gamma[\alpha, t] \subseteq \left(\tilde{f}_\lambda \Cup \tilde{g}_\gamma\right)[\alpha, t]$. □

4. APPLICATIONS TO BCK/BCI-ALGEBRAS

Definition 4.1. Let L be a BCK/BCI-algebra. A hybrid structure $\tilde{f}_\lambda$ in L over U is called a *hybrid subalgebra* of L over U if the following assertions are valid:

$$(\forall x, y \in L)\left(\begin{array}{l} \tilde{f}(x * y) \supseteq \tilde{f}(x) \cap \tilde{f}(y), \\ \lambda(x * y) \leq \bigvee\{\lambda(x), \lambda(y)\} \end{array}\right). \tag{4.1}$$

Example 4.2. Suppose that there are five houses in the initial universe set U given by

$$U = \{h_1, h_2, h_3, h_4, h_5\}.$$

Let a set of parameters $L = \{e_0, e_1, e_2, e_3\}$ be a set of status of houses in which

e_0 stands for the parameter "beautiful",
e_1 stands for the parameter "cheap",
e_2 stands for the parameter "in good location",
e_3 stands for the parameter "in green surroundings,

with the Cayley table in Table 2.

TABLE 2. Cayley table of the binary operation $*$

$*$	e_0	e_1	e_2	e_3
e_0	e_0	e_0	e_0	e_0
e_1	e_1	e_0	e_1	e_1
e_2	e_2	e_2	e_0	e_2
e_3	e_3	e_3	e_3	e_0

Then $(L, *, e_0)$ is a BCK-algebra. Let $\tilde{f}_\lambda$ be a hybrid structure in L over U which is given by Table 3.

It is routine to verify that $\tilde{f}_\lambda$ is a hybrid subalgebra of L over U.

TABLE 3. Tabular representation of the hybrid structure $\tilde{f}_\lambda$

L	$\tilde{f}$	λ
e_0	$\{h_1, h_2, h_3, h_4, h_5\}$	0.2
e_1	$\{h_1, h_4, h_5\}$	0.5
e_2	$\{h_1, h_3, h_4\}$	0.7
e_3	$\{h_2, h_3, h_5\}$	0.3

Lemma 4.1. *Every hybrid subalgebra $\tilde{f}_\lambda$ of a BCK/BCI-algebra L over U satisfies:*

$$(\forall x \in L)\left(\tilde{f}(x) \subseteq \tilde{f}(0),\ \lambda(x) \geq \lambda(0)\right). \tag{4.2}$$

Proof. Note that $x * x = 0$ for all $x \in L$. Hence

$$\tilde{f}(0) = \tilde{f}(x * x) \supseteq \tilde{f}(x) \cap \tilde{f}(x) = \tilde{f}(x),$$
$$\lambda(0) = \lambda(x * x) \leq \bigvee\{\lambda(x), \lambda(x)\} = \lambda(x)$$

for all $x \in L$. □

Proposition 4.2. *For a hybrid subalgebra $\tilde{f}_\lambda$ of a BCK/BCI-algebra L over U, the following are equivalent:*

(1) $(\forall x, y \in L)\left(\tilde{f}(x * y) \supseteq \tilde{f}(y),\ \lambda(x * y) \leq \lambda(y)\right)$.
(2) $(\forall x \in L)\left(\tilde{f}(0) = \tilde{f}(x),\ \lambda(0) = \lambda(x)\right)$.

Proof. If we take $y = 0$ in (1), then $\tilde{f}(0) \subseteq \tilde{f}(x * 0) = \tilde{f}(x)$ and $\lambda(0) \geq \lambda(x * 0) = \lambda(x)$ for all $x \in L$. Combining this and Lemma 4.1, we have $\tilde{f}(0) = \tilde{f}(x)$ and $\lambda(0) = \lambda(x)$ for all $x \in L$.

Conversely, assume that (2) is valid. Then

$$\tilde{f}(y) = \tilde{f}(0) \cap \tilde{f}(y) = \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y),$$
$$\lambda(y) = \bigvee\{\lambda(0), \lambda(y)\} = \bigvee\{\lambda(x), \lambda(y)\} \geq \lambda(x * y)$$

for all $x, y \in L$. □

Proposition 4.3. *In a BCI-algebra L, every hybrid subalgebra $\tilde{f}_\lambda$ of L over U satisfies the following condition:*

$$(\forall x, y \in L)\left(\begin{array}{l}\tilde{f}(x * (0 * y)) \supseteq \tilde{f}(x) \cap \tilde{f}(y), \\ \lambda(x * (0 * y)) \leq \bigvee\{\lambda(x), \lambda(y)\}\end{array}\right). \tag{4.3}$$

Proof. Using (4.1) and (4.2), we have

$$\tilde{f}(x * (0 * y)) \supseteq \tilde{f}(x) \cap \tilde{f}(0 * y) \supseteq \tilde{f}(x) \cap \tilde{f}(0) \cap \tilde{f}(y) = \tilde{f}(x) \cap \tilde{f}(y)$$

and

$$\lambda(x * (0 * y)) \leq \bigvee\{\lambda(x), \lambda(0 * y)\} \leq \bigvee\{\lambda(x), \bigvee\{\lambda(0), \lambda(y)\}\} = \bigvee\{\lambda(x), \lambda(y)\}$$

for all $x, y \in L$. □

Theorem 4.4. *Let L be a BCK/BCI-algebra. For a hybrid structure $\tilde{f}_\lambda$ in L over U, the following are equivalent:*

(1) *$\tilde{f}_\lambda$ is a hybrid subalgebra of L over U.*

(2) *For any* $\alpha \in \mathscr{P}(U)$ *and* $t \in I$*, the nonempty sets* $\tilde{f}_\lambda(\alpha) := \{x \in L \mid \alpha \subseteq \tilde{f}(x)\}$ *and* $\tilde{f}_\lambda(t) := \{x \in L \mid \lambda(x) \leq t\}$ *are subalgebras of* L*.*

Proof. Assume that $\tilde{f}_\lambda$ is a hybrid subalgebra of L over U. Let $\alpha \in \mathscr{P}(U)$ and $t \in I$ be such that $\tilde{f}_\lambda(\alpha) \neq \emptyset$ and $\tilde{f}_\lambda(t) \neq \emptyset$. If $x, y \in \tilde{f}_\lambda(\alpha) \cap \tilde{f}_\lambda(t)$, then $\alpha \subseteq \tilde{f}(x)$, $\alpha \subseteq \tilde{f}(y)$, $\lambda(x) \leq t$ and $\lambda(y) \leq t$. It follows from (4.1) that

$$\tilde{f}(x * y) \supseteq \tilde{f}(x) \cap \tilde{f}(y) \supseteq \alpha$$

and

$$\lambda(x * y) \leq \bigvee\{\lambda(x), \lambda(y)\} \leq t.$$

Hence $x * y \in \tilde{f}_\lambda(\alpha) \cap \tilde{f}_\lambda(t)$, and so $\tilde{f}_\lambda(\alpha)$ and $\tilde{f}_\lambda(t)$ are subalgebras of L.

Conversely, suppose that the second assertion is valid. Let $x, y \in L$ be such that $\tilde{f}(x) = \alpha_x$ and $\tilde{f}(y) = \alpha_y$. Taking $\alpha = \alpha_x \cap \alpha_y$ implies that $x, y \in \tilde{f}_\lambda(\alpha)$, and so $x * y \in \tilde{f}_\lambda(\alpha)$. Hence

$$\tilde{f}(x * y) \supseteq \alpha = \alpha_x \cap \alpha_y = \tilde{f}(x) \cap \tilde{f}(y).$$

For any $x, y \in L$, let $t := \bigvee\{\lambda(x), \lambda(y)\}$. Then $x, y \in \tilde{f}_\lambda(t)$, and so $x * y \in \tilde{f}_\lambda(t)$. It follows that

$$\lambda(x * y) \leq t = \bigvee\{\lambda(x), \lambda(y)\}.$$

Hence $\tilde{f}_\lambda$ is a hybrid subalgebra of L over U. □

Corollary 4.5. *Let* L *be a* BCK/BCI*-algebra. If* $\tilde{f}_\lambda$ *is a hybrid subalgebra of* L *over* U*, then the nonempty* $[\alpha, t]$*-hybrid cut of* $\tilde{f}_\lambda$ *is a subalgebra of* L *for all* $\alpha \in \mathscr{P}(U)$ *and* $t \in I$*.*

Proof. Straightforward. □

Theorem 4.6. *If* $\tilde{f}_\lambda$ *is a hybrid subalgebra of a* BCK/BCI*-algebra* L *over* U*, then the set*

$$\Omega := \{x \in L \mid \tilde{f}(x) \cap \alpha \neq \emptyset,\ \lambda(x) \leq t\}$$

is a subalgebra of L *for all* $(\alpha, t) \in \mathscr{P}(U) \times I$ *with* $\alpha \neq \emptyset$ *whenever it is nonempty.*

Proof. Let $(\alpha, t) \in \mathscr{P}(U) \times I$ be such that $\Omega \neq \emptyset \neq \alpha$. Let $x, y \in \Omega$. Then $\tilde{f}(x) \cap \alpha \neq \emptyset \neq \tilde{f}(y) \cap \alpha$, $\lambda(x) \leq t$ and $\lambda(y) \leq t$. It follows from (4.1) that

$$\begin{aligned} \tilde{f}(x * y) \cap \alpha &\supseteq (\tilde{f}(x) \cap \tilde{f}(y)) \cap \alpha \\ &= (\tilde{f}(x) \cap \alpha) \cap (\tilde{f}(y) \cap \alpha) \neq \emptyset \end{aligned}$$

and $\lambda(x * y) \leq \bigvee\{\lambda(x), \lambda(y)\} \leq t$. Hence $x * y \in \Omega$, and so Ω is a subalgebra of L. □

Theorem 4.7. *Let* L *be a* BCK/BCI*-algebra. If* $\tilde{f}_\lambda$ *and* $\tilde{g}_\gamma$ *are hybrid subalgebras of* L *over* U*, then so is the hybrid intersection* $\tilde{f}_\lambda \Cap \tilde{g}_\gamma$*.*

Proof. For any $x, y \in L$, we have

$$\begin{aligned} &\left(\tilde{f} \tilde{\cap} \tilde{g}\right)(x * y) = \tilde{f}(x * y) \cap \tilde{g}(x * y) \supseteq \left(\tilde{f}(x) \cap \tilde{f}(y)\right) \cap (\tilde{g}(x) \cap \tilde{g}(y)) \\ &= \left(\tilde{f}(x) \cap \tilde{g}(x)\right) \cap \left(\tilde{f}(y) \cap \tilde{g}(y)\right) = \left(\tilde{f} \tilde{\cap} \tilde{g}\right)(x) \cap \left(\tilde{f} \tilde{\cap} \tilde{g}\right)(y) \end{aligned}$$

and

$$\begin{aligned}(\lambda \vee \gamma)(x * y) &= \bigvee \{\lambda(x * y), \gamma(x * y)\} \\ &\leq \bigvee \left\{ \bigvee \{\lambda(x), \lambda(y)\}, \bigvee \{\gamma(x), \gamma(y)\} \right\} \\ &= \bigvee \left\{ \bigvee \{\lambda(x), \gamma(x)\}, \bigvee \{\lambda(y), \gamma(y)\} \right\} \\ &= \bigvee \{(\lambda \vee \gamma)(x), (\lambda \vee \gamma)(y)\}.\end{aligned}$$

Hence $\tilde{f}_\lambda \Cap \tilde{g}_\gamma$ is a hybrid subalgebra of L over U. □

The following example shows that the hybrid union of hybrid subalgebras may not be a hybrid subalgebra.

Example 4.3. Suppose that there are five patients in a hospital given by

$$U = \{p_1, p_2, p_3, p_4, p_5\}.$$

As a set of parameters, we consider $L = \{c, e, g, h, t\}$ which is a set of status of patients where

c stands for the parameter "cough",
e stands for the parameter "eye disease",
g stands for the parameter "gastric cancer",
h stands for the parameter "headache",
t stands for the parameter "toothache".

We define a binary operation on L by the Cayley table in Table 4.

TABLE 4. Cayley table of the binary operation $*$

$*$	c	e	g	h	t
c	c	c	c	h	h
e	e	c	e	t	h
g	g	g	c	h	h
h	h	h	h	c	c
t	t	h	t	e	c

Then $(L, *, c)$ is a BCI-algebra. Let $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ be hybrid structures in L over U which are given by Table 5 and Table 6, respectively. Then the hybrid union $\tilde{f}_\lambda \Cup \tilde{g}_\gamma$ of $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ is represented by the by Table 7.

TABLE 5. Tabular representation of the hybrid structure $\tilde{f}_\lambda$

L	$\tilde{f}$	λ
c	U	0.4
e	$\{p_1, p_2, p_3, p_4\}$	0.4
g	$\{p_2, p_4\}$	0.7
h	$\{p_2, p_4\}$	0.7
t	$\{p_2, p_4\}$	0.7

TABLE 6. Tabular representation of the hybrid structure $\tilde{g}_\gamma$

L	$\tilde{g}$	γ
c	U	0.2
e	$\{p_2, p_4\}$	0.8
g	$\{p_1, p_2, p_3, p_4\}$	0.4
h	$\{p_2, p_3, p_4\}$	0.6
t	$\{p_2, p_4\}$	0.8

TABLE 7. Tabular representation of the hybrid structure $\tilde{f}_\lambda \Cup \tilde{g}_\gamma$

L	$\tilde{f} \tilde{\cup} \tilde{g}$	$\lambda \wedge \gamma$
c	U	0.2
e	$\{p_1, p_2, p_3, p_4\}$	0.4
g	$\{p_1, p_2, p_3, p_4\}$	0.4
h	$\{p_2, p_3, p_4\}$	0.6
t	$\{p_2, p_4\}$	0.7

By routine calculations, we know that $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ are hybrid subalgebras of L over U. But, the hybrid union $\tilde{f}_\lambda \Cup \tilde{g}_\gamma$ is not a hybrid subalgebra of L over U since

$$\left(\tilde{f} \tilde{\cup} \tilde{g}\right)(e * h) = \left(\tilde{f} \tilde{\cup} \tilde{g}\right)(t) = \{p_2, p_4\} \not\supseteq \{p_2, p_3, p_4\} = \left(\tilde{f} \tilde{\cup} \tilde{g}\right)(e) \cap \left(\tilde{f} \tilde{\cup} \tilde{g}\right)(h)$$

and/or

$$(\lambda \wedge \gamma)(e * h) = (\lambda \wedge \gamma)(t) = 0.7 \not\leq 0.6 = \bigvee \{(\lambda \wedge \gamma)(e), (\lambda \wedge \gamma)(h)\}.$$

For any hybrid structure $\tilde{f}_\lambda$ in L over U, let $\tilde{f}_\lambda^* := \left(\tilde{f}^*, \lambda^*\right)$ be a hybrid structure in L over U defined by

$$\tilde{f}^* : L \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tilde{f}(x) & \text{if } x \in \tilde{f}_\lambda(\alpha), \\ \beta & \text{otherwise,} \end{cases}$$

$$\lambda^* : L \to I,\ x \mapsto \begin{cases} \lambda(x) & \text{if } x \in \tilde{f}_\lambda(t), \\ s & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in \mathscr{P}(U)$ and $s, t \in I$ with $\beta \subsetneq \tilde{f}(x)$ and $s > \lambda(x)$.

Theorem 4.8. *Let L be a BCK/BCI-algebra. If $\tilde{f}_\lambda$ is a hybrid subalgebra of L over U, then so is $\tilde{f}_\lambda^*$.*

Proof. Assume that $\tilde{f}_\lambda$ is a hybrid subalgebra of a BCK/BCI-algebra L over U. Then $\tilde{f}_\lambda(\alpha)$ and $\tilde{f}_\lambda(t)$ are subalgebras of L for all $\alpha \in \mathscr{P}(U)$ and $t \in I$ provided that they are nonempty by Theorem 4.4. Let $x, y \in L$. If $x, y \in \tilde{f}_\lambda(\alpha)$, then $x * y \in \tilde{f}_\lambda(\alpha)$. Thus

$$\tilde{f}^*(x * y) = \tilde{f}(x * y) \supseteq \tilde{f}(x) \cap \tilde{f}(y) = \tilde{f}^*(x) \cap \tilde{f}^*(y).$$

If $x \notin \tilde{f}_\lambda(\alpha)$ or $y \notin \tilde{f}_\lambda(\alpha)$, then $\tilde{f}^*(x) = \beta$ or $\tilde{f}^*(y) = \beta$. Hence

$$\tilde{f}^*(x * y) \supseteq \beta = \tilde{f}^*(x) \cap \tilde{f}^*(y).$$

Now, if $x, y \in \tilde{f}_\lambda(t)$, then $x * y \in \tilde{f}_\lambda(t)$. Thus

$$\lambda^*(x * y) = \lambda(x * y) \leq \bigvee\{\lambda(x), \lambda(y)\} = \bigvee \{\lambda^*(x), \lambda^*(y)\}.$$

If $x \notin \tilde{f}_\lambda(t)$ or $y \notin \tilde{f}_\lambda(t)$, then $\lambda^*(x) = s$ or $\lambda^*(y) = s$. Hence

$$\lambda^*(x * y) \leq s = \bigvee \{\lambda^*(x), \lambda^*(y)\}.$$

Therefore $\tilde{f}^*_\lambda$ is a hybrid subalgebra of L over U. □

The following example shows that the converse of Theorem 4.8 is not true in general.

Example 4.4. Suppose that there are ten houses in the initial universe set U given by

$$U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}\}.$$

Let a set of parameters $L = \{e_0, e_1, e_2, e_3\}$ be a set of status of houses in which

e_0 stands for the parameter "beautiful",
e_1 stands for the parameter "cheap",
e_2 stands for the parameter "in good location",
e_3 stands for the parameter "in green surroundings,

with the Cayley table in Table 8.

TABLE 8. Cayley table of the binary operation $*$

$*$	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_0	e_3	e_2
e_2	e_2	e_3	e_0	e_1
e_3	e_3	e_2	e_1	e_0

Then $(L, *, e_0)$ is a BCI-algebra. Let $\tilde{f}_\lambda$ be a hybrid structure in L over U which is given by Table 9.

TABLE 9. Tabular representation of the hybrid structure $\tilde{f}_\lambda$

L	$\tilde{f}$	λ
e_0	U	0.3
e_1	$\{h_2, h_4, h_6, h_8, h_{10}\}$	0.7
e_2	$\{h_3, h_6, h_9\}$	0.9
e_3	$\{h_8\}$	0.9

Then $\tilde{f}_\lambda(\alpha) = \{e_0, e_1\}$ for $\alpha = \{h_2, h_4, h_6, h_8, h_{10}\}$, and $\tilde{f}_\lambda(t) = \{e_0, e_1\}$ for $t = 0.7$. Let $\tilde{f}^*_\lambda := \left(\tilde{f}^*, \lambda^*\right)$ be a hybrid structure in L over U defined by

$$\tilde{f}^* : L \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tilde{f}(x) & \text{if } x \in \tilde{f}_\lambda(\alpha), \\ \emptyset & \text{otherwise,} \end{cases}$$

$$\lambda^* : L \to I,\ x \mapsto \begin{cases} \lambda(x) & \text{if } x \in \tilde{f}_\lambda(t), \\ 1 & \text{otherwise,} \end{cases}$$

that is,

$$\tilde{f}^* : L \to \mathscr{P}(U),\ x \mapsto \begin{cases} U & \text{if } x = e_0, \\ \{h_2, h_4, h_6, h_8, h_{10}\} & \text{if } x = e_1, \\ \emptyset & \text{if } x \in \{e_2, e_3\}, \end{cases}$$

$$\lambda^* : L \to I,\ x \mapsto \begin{cases} 0.3 & \text{if } x = e_0, \\ 0.7 & \text{if } x = e_1, \\ 1 & \text{if } x \in \{e_2, e_3\}. \end{cases}$$

It is routine to verify that $\tilde{f}^*_\lambda := \left(\tilde{f}^*, \lambda^*\right)$ is a hybrid subalgebra of L over U. But $\tilde{f}_\lambda$ is not a hybrid subalgebra of L over U since

$$\tilde{f}(e_1) \cap \tilde{f}(e_2) = \{h_6\} \nsubseteq \{h_8\} = \tilde{f}(e_3) = \tilde{f}(e_1 * e_2).$$

Let $\theta : L \to M$ be a mapping from a set L to a set M. For a hybrid structure $\tilde{g}_\gamma$ in M over U, consider a hybrid structure $\theta^{-1}(\tilde{g}_\gamma) := \left(\theta^{-1}(\tilde{g}), \theta^{-1}(\gamma)\right)$ in L over U where $\theta^{-1}(\tilde{g})(x) = \tilde{g}(\theta(x))$ and $\theta^{-1}(\gamma)(x) = \gamma(\theta(x))$ for all $x \in L$. We say that $\theta^{-1}(\tilde{g}_\gamma)$ is the *hybrid preimage* of $\tilde{g}_\gamma$ under θ. For a hybrid structure $\tilde{f}_\lambda$ in L over U, the *hybrid image* of $\tilde{f}_\lambda$ under θ is defined to be a hybrid structure $\theta(\tilde{f}_\lambda) := \left(\theta(\tilde{f}), \theta(\lambda)\right)$ in M over U where

$$\theta(\tilde{f})(y) = \begin{cases} \bigcup\limits_{x \in \theta^{-1}(y)} \tilde{f}(x) & \text{if } \theta^{-1}(y) \neq \emptyset, \\ \emptyset & \text{otherwise}, \end{cases}$$

$$\theta(\lambda)(y) = \begin{cases} \bigwedge\limits_{x \in \theta^{-1}(y)} \lambda(x) & \text{if } \theta^{-1}(y) \neq \emptyset, \\ 1 & \text{otherwise}, \end{cases}$$

for every $y \in M$.

Theorem 4.9. *Every homomorphic hybrid preimage of a hybrid subalgebra is also a hybrid subalgebra.*

Proof. Let $\theta : L \to M$ be a homomorphism of BCK/BCI-algebras. Let $\tilde{g}_\gamma$ is a hybrid subalgebra of M over U and let $x, y \in L$. Then

$$\begin{aligned} \theta^{-1}(\tilde{g})(x * y) &= \tilde{g}(\theta(x * y)) = \tilde{g}(\theta(x) * \theta(y)) \\ &\supseteq \tilde{g}(\theta(x)) \cap \tilde{g}(\theta(y)) \\ &= \theta^{-1}(\tilde{g})(x) \cap \theta^{-1}(\tilde{g})(y) \end{aligned}$$

and

$$\begin{aligned} \theta^{-1}(\gamma)(x * y) &= \gamma(\theta(x * y)) = \gamma(\theta(x) * \theta(y)) \\ &\leq \bigvee\{\gamma(\theta(x)), \gamma(\theta(y))\} \\ &= \bigvee\{\theta^{-1}(\gamma)(x), \theta^{-1}(\gamma)(y)\}. \end{aligned}$$

Therefore $\theta^{-1}(\tilde{g}_\gamma)$ is a hybrid subalgebra of L over U. □

For an onto homomorphism $\theta : L \to M$ of BCK/BCI-algebras, let $\theta^{-1}(\tilde{g}_\gamma) := \left(\theta^{-1}(\tilde{g}), \theta^{-1}(\gamma)\right)$ be a hybrid subalgebra of L over U where $\tilde{g}_\gamma$ is a hybrid structure in M over U. Let $a, b \in M$. Then $\theta(x) = a$ and $\theta(y) = b$ for some $x, y \in L$ since θ is onto.

Hence

$$\begin{aligned}\tilde{g}(a*b) &= \tilde{g}(\theta(x)*\theta(y)) = \tilde{g}(\theta(x*y)) = \theta^{-1}(\tilde{g})(x*y)\\ &\supseteq \theta^{-1}(\tilde{g})(x) \cap \theta^{-1}(\tilde{g})(y) = \tilde{g}(\theta(x)) \cap \tilde{g}(\theta(y)) = \tilde{g}(a) \cap \tilde{g}(b)\end{aligned}$$

and

$$\begin{aligned}\gamma(a*b) &= \gamma(\theta(x)*\theta(y)) = \gamma(\theta(x*y)) = \theta^{-1}(\gamma)(x*y)\\ &\leq \bigvee\{\theta^{-1}(\gamma)(x), \theta^{-1}(\gamma)(y)\} = \bigvee\{\gamma(\theta(x)), \gamma(\theta(y))\}\\ &= \bigvee\{\gamma(a), \gamma(b)\}.\end{aligned}$$

Therefore we have the following theorem.

Theorem 4.10. *Let $\theta : L \to M$ be an onto homomorphism of BCK/BCI-algebras. For every hybrid structure $\tilde{g}_\gamma$ in M over U, if the preimage $\theta^{-1}(\tilde{g}_\gamma)$ of $\tilde{g}_\gamma$ under θ is a hybrid subalgebra of L over U, then $\tilde{g}_\gamma$ is a hybrid subalgebra of M over U.*

5. Applications to linear spaces

Definition 5.1. Let L be a field. A hybrid structure $\tilde{f}_\lambda$ in L over U is called a *hybrid field* of L over U if

$$(\forall a, b \in L)\left(\tilde{f}(a+b) \supseteq \tilde{f}(a) \cap \tilde{f}(b),\ \lambda(a+b) \leq \bigvee\{\lambda(a), \lambda(b)\}\right), \quad (5.1)$$

$$(\forall a \in L)\left(\tilde{f}(-a) \supseteq \tilde{f}(a),\ \lambda(-a) \leq \lambda(a)\right), \quad (5.2)$$

$$(\forall a, b \in L)\left(\tilde{f}(ab) \supseteq \tilde{f}(a) \cap \tilde{f}(b),\ \lambda(ab) \leq \bigvee\{\lambda(a), \lambda(b)\}\right), \quad (5.3)$$

$$(\forall a \in L)\left(a \neq 0 \ \Rightarrow\ \tilde{f}(a^{-1}) \supseteq \tilde{f}(a),\ \lambda(a^{-1}) \leq \lambda(a)\right). \quad (5.4)$$

Proposition 5.1. *If $\tilde{f}_\lambda$ is a hybrid field of a field L over U, then*

(i) $(\forall a \in L)\left(\tilde{f}(a) \subseteq \tilde{f}(0),\ \lambda(a) \geq \lambda(0)\right)$,

(ii) $(\forall a \in L)\left(a \neq 0 \ \Rightarrow\ \tilde{f}(a) \subseteq \tilde{f}(1),\ \lambda(a) \geq \lambda(1)\right)$,

(iii) $\tilde{f}(1) \subseteq \tilde{f}(0),\ \lambda(1) \geq \lambda(0)$.

Proof. (i) For every $a \in L$, we have $\tilde{f}(0) = \tilde{f}(a + (-a)) \supseteq \tilde{f}(a) \cap \tilde{f}(-a) = \tilde{f}(a)$ and $\lambda(0) = \lambda(a + (-a)) \leq \bigvee\{\lambda(a), \lambda(-a)\} = \lambda(a)$.

(ii) Let $a \in L$ and $a \neq 0$. Then $\tilde{f}(1) = \tilde{f}(aa^{-1}) \supseteq \tilde{f}(a) \cap \tilde{f}(a^{-1}) = \tilde{f}(a)$ and $\lambda(1) = \lambda(aa^{-1}) \leq \bigvee\{\lambda(a), \lambda(a^{-1})\} = \lambda(a)$.

(iii) It is by (i). □

Definition 5.2. Let $\tilde{f}_\lambda$ be a hybrid field of a field L over U, and let Y be a linear space over L. A hybrid structure $\tilde{g}_\gamma$ in Y over U is called a *hybrid linear space* over $(\tilde{f}_\lambda, L)$ if the following conditions hold.

$$(\forall x, y \in Y)\left(\tilde{g}(x+y) \supseteq \tilde{g}(x) \cap \tilde{g}(y),\ \gamma(x+y) \leq \bigvee\{\gamma(x), \gamma(y)\}\right), \quad (5.5)$$

$$(\forall x \in Y)\left(\tilde{g}(-x) \supseteq \tilde{g}(x),\ \gamma(-x) \leq \gamma(x)\right), \quad (5.6)$$

$$(\forall a \in L)(\forall x \in Y)\left(\tilde{g}(ax) \supseteq \tilde{f}(a) \cap \tilde{g}(x),\ \gamma(ax) \leq \bigvee\{\lambda(a), \gamma(x)\}\right), \quad (5.7)$$

$$\tilde{f}(1) \supseteq \tilde{g}(0),\ \lambda(1) \leq \gamma(0). \quad (5.8)$$

Proposition 5.2. *If $\tilde{g}_\gamma$ is a hybrid linear space over $(\tilde{f}_\lambda, L)$, then*

(i) $\tilde{f}(0) \supseteq \tilde{g}(0),\ \lambda(0) \leq \gamma(0)$,
(ii) $(\forall x \in Y)\ (\tilde{g}(0) \supseteq \tilde{g}(x),\ \gamma(0) \leq \gamma(x))$,
(iii) $(\forall x \in Y)\ \Big(\tilde{f}(0) \supseteq \tilde{g}(x),\ \lambda(0) \leq \gamma(x)\Big)$.

Proof. Straightforward. □

Theorem 5.3. *Let $\tilde{f}_\lambda$ be a hybrid field of a field L over U, and let Y be a linear space over L. A hybrid structure $\tilde{g}_\gamma$ in Y over U is a hybrid linear space over $(\tilde{f}_\lambda, L)$ if and only if*

(i) $\tilde{g}(ax+by) \supseteq \tilde{f}(a)\cap\tilde{g}(x)\cap\tilde{f}(b)\cap\tilde{g}(y)$ *and* $\gamma(ax+by) \leq \bigvee\{\lambda(a),\gamma(x),\lambda(b),\gamma(y)\}$
(ii) $\tilde{f}(1) \supseteq \tilde{g}(x)$ *and* $\lambda(1) \leq \gamma(x)$

for all $x, y \in Y$ and $a, b \in L$.

Proof. Assume that $\tilde{g}_\gamma$ is a hybrid linear space over $(\tilde{f}_\lambda, L)$. Let $a, b \in L$ and $x, y \in Y$. Then

$$\begin{aligned}\tilde{g}(ax+by) &\supseteq \tilde{g}(ax)\cap\tilde{g}(by) \supseteq (\tilde{f}(a)\cap\tilde{g}(x))\cap(\tilde{f}(b)\cap\tilde{g}(y))\\ &= \tilde{f}(a)\cap\tilde{g}(x)\cap\tilde{f}(b)\cap\tilde{g}(y)\end{aligned}$$

and

$$\begin{aligned}\gamma(ax+by) &\leq \bigvee\{\gamma(ax),\gamma(by)\} \leq \bigvee\Big\{\bigvee\{\lambda(a),\gamma(x)\},\bigvee\{\lambda(b),\gamma(y)\}\Big\}\\ &= \bigvee\{\lambda(a),\gamma(x),\lambda(b),\gamma(y)\}.\end{aligned}$$

The second result is induced by (5.8) and Proposition 5.2(ii).

Conversely, suppose that (i) and (ii) are valid. Then

$$\tilde{g}(x+y) = \tilde{g}(1x+1y) \supseteq \tilde{f}(1)\cap\tilde{g}(x)\cap\tilde{f}(1)\cap\tilde{g}(y) = \tilde{g}(x)\cap\tilde{g}(y)$$

and

$$\gamma(x+y) = \gamma(1x+1y) \leq \bigvee\{\lambda(1),\gamma(x),\lambda(1),\gamma(y)\} = \bigvee\{\gamma(x),\gamma(y)\}$$

for all $x, y \in Y$. Since $\tilde{f}_\lambda$ is a hybrid field of L over U,

$$\begin{aligned}&\tilde{f}(0) \supseteq \tilde{f}(1) \supseteq \tilde{g}(x),\ \lambda(0) \leq \lambda(1) \leq \gamma(x),\\ &\tilde{f}(-1) \supseteq \tilde{f}(1) \supseteq \tilde{g}(x),\ \lambda(-1) \leq \lambda(1) \leq \gamma(x)\end{aligned}$$

for all $x \in Y$. Hence

$$\begin{aligned}\tilde{g}(-x) &= \tilde{g}(0x+(-1)x) \supseteq \tilde{f}(0)\cap\tilde{g}(x)\cap\tilde{f}(-1)\cap\tilde{g}(x)\\ &= \tilde{g}(x)\cap\tilde{g}(x) = \tilde{g}(x)\end{aligned}$$

and

$$\begin{aligned}\gamma(-x) &= \gamma(0x+(-1)x) \leq \bigvee\{\lambda(0),\gamma(x),\lambda(-1),\gamma(x)\}\\ &= \bigvee\{\gamma(x),\gamma(x)\} = \gamma(x).\end{aligned}$$

For any $a \in L$ and $x \in Y$, we have

$$\tilde{g}(ax) = \tilde{g}(0x+ax) \supseteq \tilde{f}(0)\cap\tilde{g}(x)\cap\tilde{f}(a)\cap\tilde{g}(x) = \tilde{f}(a)\cap\tilde{g}(x)$$

and

$$\gamma(ax) = \gamma(0x+ax) \leq \bigvee\{\lambda(0),\gamma(x),\lambda(a),\gamma(x)\} = \bigvee\{\lambda(a),\gamma(x)\}.$$

Obviously, $\tilde{f}(1) \supseteq \tilde{g}(0)$ and $\lambda(1) \leq \gamma(0)$. Therefore $\tilde{g}_\gamma$ is a hybrid linear space over $(\tilde{f}_\lambda, L)$. □

Theorem 5.4. *Let θ be a linear transformation of Y into Z where Y and Z are linear spaces over a field L. If $\tilde{f}_\lambda$ is a hybrid field of L over U and $\tilde{h}_\tau$ is a hybrid linear space over $(\tilde{f}_\lambda, L)$ in Z, then $\theta^{-1}(\tilde{h}_\tau) := \Big(\theta^{-1}(\tilde{h}), \theta^{-1}(\tau)\Big)$ is a hybrid linear space over $(\tilde{f}_\lambda, L)$ in Y.*

Proof. Let $x, y \in Y$ and $a, b \in L$. Then

$$\begin{aligned}\theta^{-1}(\tilde{h})(ax+by) &= \tilde{h}(\theta(ax+by)) = \tilde{h}(a\theta(x)+b\theta(y)) \\ &\supseteq \tilde{f}(a) \cap \tilde{h}(\theta(x)) \cap \tilde{f}(b) \cap \tilde{h}(\theta(y)) \\ &= \tilde{f}(a) \cap \theta^{-1}(\tilde{h})(x) \cap \tilde{f}(b) \cap \theta^{-1}(\tilde{h})(y)\end{aligned}$$

and

$$\begin{aligned}\theta^{-1}(\tau)(ax+by) &= \tau(\theta(ax+by)) = \tau(a\theta(x)+b\theta(y)) \\ &\leq \bigvee \{\lambda(a), \tau(\theta(x)), \lambda(b), \tau(\theta(y))\} \\ &= \bigvee \left\{\lambda(a), \theta^{-1}(\tau)(x), \lambda(b), \theta^{-1}(\tau)(y)\right\}.\end{aligned}$$

Obviously, $\tilde{f}(1) \supseteq \theta^{-1}(\tilde{h})(x)$ and $\lambda(1) \leq \theta^{-1}(\tau)(x)$ for all $x \in Y$. Therefore $\theta^{-1}(\tilde{h}_\tau) := \Big(\theta^{-1}(\tilde{h}), \theta^{-1}(\tau)\Big)$ is a hybrid linear space over $(\tilde{f}_\lambda, L)$ in Y by Theorem 5.3. □

Theorem 5.5. *Let θ be a linear transformation of Y into Z where Y and Z are linear spaces over a field L. If $\tilde{f}_\lambda$ is a hybrid field of L over U and $\tilde{g}_\gamma$ is a hybrid linear space over $(\tilde{f}_\lambda, L)$ in Y, then $\theta(\tilde{g}_\gamma)$ is a hybrid linear space over $(\tilde{f}_\lambda, L)$ in Z.*

Proof. Let $a, b \in L$ and $u, v \in Z$. If $\theta^{-1}(u) = \emptyset$ or $\theta^{-1}(v) = \emptyset$, then the condition (i) of Theorem 5.3 is satisfied. Assume that $\theta^{-1}(u) \neq \emptyset \neq \theta^{-1}(v)$. Then $\theta^{-1}(au+bv) \neq \emptyset$. Let $r \in \theta^{-1}(u)$ and $s \in \theta^{-1}(v)$. Then $\theta(ar+bs) = a\theta(r) + b\theta(s) = au+bv$, and so

$$\begin{aligned}\theta(\tilde{g})(au+bv) &= \bigcup_{w \in \theta^{-1}(au+bv)} \tilde{g}(w) \\ &\supseteq \bigcup_{r \in \theta^{-1}(u),\ s \in \theta^{-1}(v)} \tilde{g}(ar+bs) \\ &\supseteq \bigcup_{r \in \theta^{-1}(u),\ s \in \theta^{-1}(v)} \Big(\tilde{f}(a) \cap \tilde{g}(r) \cap \tilde{f}(b) \cap \tilde{g}(s)\Big) \\ &= \left(\tilde{f}(a) \cap \bigcup_{r \in \theta^{-1}(u)} \tilde{g}(r)\right) \cap \left(\tilde{f}(b) \cap \bigcup_{s \in \theta^{-1}(v)} \tilde{g}(s)\right) \\ &= \Big(\tilde{f}(a) \cap \theta(\tilde{g})(u)\Big) \cap \Big(\tilde{f}(b) \cap \theta(\tilde{g})(v)\Big) \\ &= \tilde{f}(a) \cap \theta(\tilde{g})(u) \cap \tilde{f}(b) \cap \theta(\tilde{g})(v)\end{aligned}$$

and

$$\begin{aligned}\theta(\gamma)(au+bv) &= \bigwedge_{w\in\theta^{-1}(au+bv)} \gamma(w) \\ &\leq \bigwedge_{r\in\theta^{-1}(u),\ s\in\theta^{-1}(v)} \gamma(ar+bs) \\ &\leq \bigwedge_{r\in\theta^{-1}(u),\ s\in\theta^{-1}(v)} \left(\bigvee\{\lambda(a),\gamma(r),\lambda(b),\gamma(s)\}\right) \\ &= \bigvee\left\{\bigvee\left\{\lambda(a), \bigwedge_{r\in\theta^{-1}(u)}\gamma(r)\right\}, \bigvee\left\{\lambda(b), \bigwedge_{s\in\theta^{-1}(v)}\gamma(s)\right\}\right\} \\ &= \bigvee\{\lambda(a),\theta(\gamma)(u),\lambda(b),\theta(\gamma)(v)\}.\end{aligned}$$

Obviously $\tilde{f}(1) \supseteq \theta(\tilde{g})(x)$ and $\lambda(1) \leq \theta(\gamma)(x)$ for all $x \in Z$. It follows from Theorem 5.3 that $\theta(\tilde{g}_\gamma)$ is a hybrid linear space over over $(\tilde{f}_\lambda, L)$ in Z. □

6. Conclusion

In this paper, we introduced the notion of hybrid structure in a set of parameters over an initial universe set, and investigate several properties. Using this notion, we introduce the concepts of a hybrid subalgebra, a hybrid field and a hybrid linear space. We consider the hybrid union and hybrid intersection of hybrid subalgebras in BCK/BCI-algebras. We discuss characterizations of a hybrid subalgebra and a hybrid linear space. Given a hybrid subalgebra, we make a new hybrid subalgebra in BCK/BCI-algebras. We consider the hybrid image and preimage of a hybrid subalgebra and a hybrid linear space under the BCK/BCI-homomorphisms and the linear transformation of linear spaces, respectively.

Work is ongoing. Some important issues for future work are: (1) to develop strategies for obtaining more valuable results, (2) to apply these notions and results for studying related notions in other (hyper) algebraic structures, (3) to extend these results, one can further study the hybrid algebraic structures on different algebras such as MTL-algerbas, BL-algebras, MV-algebras, EQ-algebras, R0-algebras and Q-algebras etc.

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