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PYTHAGOREAN CUBIC IDEAL IN SEMIGROUP

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ABSTRACT. In this paper, we introduce the notion of Pythagorean cubic ideal in semigroup. Also, we discuss some of their properties with examples.

1. INTRODUCTION

In 1965, Zadeh[8, 9] presented the idea of a fuzzy set. He also developed the notion of interval-valued fuzzy set in 1975 which is an expansion of the fuzzy set. A semigroup is an algebraic structure comprising of a non-empty set along with an affiliated bi- nary operation. Atanassvo[2] presented the intuitionistic fuzzy set with certain properties. Atanassvo et al.[3] developed the idea of interval-valued intuitionistic fuzzy set. In 2012, Jun et al.[5] presented the idea of cubic set a combination of interval-valued fuzzy set and fuzzy set and talked about some related properties. Afterward, in 2013, Jun and Khan[6] presented the idea of cubic ideals in the semigroup. In 2013, Yager[**?**] started the idea of Pythagorean fuzzy set, the sum of the squares degree of membership(DOM) and degree of non-membership(DONM) has a place with the unit interval [0,1]. In 2019, Abbas et al.[1] introduced Cubic Pythagorean fuzzy sets. In 2019, Hussain et al.[4] started the ideas of Rough Pythagorean fuzzy ideals in the semigroup. In this paper, we introduce the properties of Pythagorean cubic ideals in semigroup.

2. PRELIMINARIES

Definition 2.1. [**?**] Let X be a universe of discourse, A **Pythagorean fuzzy set** (PFS) $P = \{w, \phi_p(w), \psi_p(w)/w \in X\}$ where $\phi : X \to [0,1]$ and $\psi : X \to [0,1]$ represent the DOM and DONM of the object $w \in X$ to the set P subset to the condition $0 \leq (\phi_p(w))^2 + (\psi_p(w))^2 \leq 1$ for all $w \in X$. For the sake of simplicity a PFS is denoted as $P = (\phi_p(w), \psi_p(w))$.

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3. Pythagorean Cubic ideal in semigroup

Definition 3.1. A Pythagorean cubic set(PCS) $P^c = (\phi_p^c, \psi_p^c) = \left\langle \left[\widetilde{\phi_p}, \widetilde{\psi_p}\right], (\phi_p, \psi_p) \right\rangle$ on S is known to be a Pythagorean cubic sub-semigroup(PFSS) of S. If for all $w_1, w_2 \in S$, it holds
$\widetilde{\phi_p}(w_1 w_2) \geq min\left\{\widetilde{\phi_p}(w_1), \widetilde{\phi_p}(w_2)\right\}$
$\widetilde{\psi_p}(w_1 w_2) \leq max\left\{\widetilde{\psi_p}(w_1), \widetilde{\psi_p}(w_2)\right\}$
$\phi_p(w_1 w_2) \leq max\{\phi_p(w_1), \phi_p(w_2)\}$
$\psi_p(w_1 w_2) \geq min\{\psi_p(w_1), \psi_p(w_2)\}$.

Consider

TABLE 1. Cayley table

$\bullet$	u	v	w	x	y
u	u	u	u	u	u
v	u	v	u	x	u
w	u	y	w	w	y
x	u	v	x	x	v
y	u	y	u	w	u

Example 3.2. Consider a semigroup $S = \{u, v, w, x, y\}$ with the above Cayley Table. Define a Pythagorean cubic set(PCS) $P^c = \left\langle \left[\widetilde{\phi_p}, \widetilde{\psi_p}\right], (\phi_p, \psi_p) \right\rangle$ in S as follows.

S	$\left[\widetilde{\phi_p}(w_1), \widetilde{\psi_p}(w_1)\right]$	$(\phi_p(w_1), \psi_p(w_1))$
u	$[0.7, 0.8], [0.1, 0.2]$	$0.2, 0.7$
v	$[0.4, 0.6], [0.4, 0.5]$	$0.4, 0.5$
w	$[0.3, 0.5], [0.5, 0.6]$	$0.5, 0.3$
x	$[0.1, 0.2], [0.3, 0.5]$	$0.3, 0.2$
y	$[0.3, 0.5], [0.5, 0.6]$	$0.5, 0.3$

Thus $P^c = \left\langle \left[\widetilde{\phi_p}, \widetilde{\psi_p}\right], (\phi_p, \psi_p) \right\rangle$ is a PCSS of S.

Definition 3.3. A PCS $P^c = (\phi_{p^c}, \psi_{p^c})$ on semigroup S, is said to be a PCL (P^c_{LI})(resp.PCR(P^c_{RI})) ideal of S. If for all $w_1, w_2 \in S$, it holds.
$\widetilde{\phi_p}(w_1 w_2) \geq \widetilde{\phi_p}(w_2)$; $\phi_p(w_1 w_2) \leq \phi_p(w_2)$
$\widetilde{\psi_p}(w_1 w_2) \leq \widetilde{\psi_p}(w_2)$; $\psi_p(w_1 w_2) \geq \psi_p(w_2)$
resp.right(P^c_{RI})
$\widetilde{\phi_p}(w_1 w_2) \geq \widetilde{\phi_p}(w_1)$; $\phi_p(w_1 w_2) \leq \phi_p(w_1)$
$\widetilde{\psi_p}(w_1 w_2) \leq \widetilde{\psi_p}(w_1)$; $\psi_p(w_1 w_2) \geq \psi_p(w_1)$.

Definition 3.4. A PCS $P^c = (\phi_{p^c}, \psi_{p^c}) = \left\langle \left[\widetilde{\phi_p}, \widetilde{\psi_p}\right], (\phi_p, \psi_p) \right\rangle$ on S is called (PCI)Pythagorean cubic ideal $(P^c_I)$of S. If for all $w_1, w_2 \in S$, it P^c is both a left and right (PCI)Pythagorean cubic ideal of S.
$\widetilde{\phi_p}(w_1 w_2) \geq max\left\{\widetilde{\phi_p}(w_1), \widetilde{\phi_p}(w_2)\right\}$
$\widetilde{\psi_p}(w_1 w_2) \leq min\left\{\widetilde{\psi_p}(w_1), \widetilde{\psi_p}(w_2)\right\}$
$\phi_p(w_1 w_2) \leq min\{\phi_p(w_1), \phi_p(w_2)\}$
$\psi_p(w_1 w_2) \geq max\{\psi_p(w_1), \psi_p(w_2)\}$.

Definition 3.5. A PCS $P^c = (\phi_{p^c}, \psi_{p^c}) = \left\langle \left[\widetilde{\phi_p}, \widetilde{\psi_p}\right], (\phi_p, \psi_p) \right\rangle$ on S is known to be a (PCBI)Pythagorean cubic Bi-ideal (P^c_{BI}) of S. If for all $w_1, w_2, w_3 \in S$, and satisfy.
$\widetilde{\phi_p}\left(w_1 w_2 w_3\right) \geq min\left\{\widetilde{\phi_p}(w_1), \widetilde{\phi_p}(w_3)\right\}$
$\widetilde{\psi_p}\left(w_1 w_2 w_3\right) \leq max\left\{\widetilde{\psi_p}(w_1), \widetilde{\psi_p}(w_3)\right\}$
$\phi_p\left(w_1 w_2 w_3\right) \leq max\left\{\phi_p(w_1), \phi_p(w_3)\right\}$
$\psi_p\left(w_1 w_2 w_3\right) \geq min\left\{\psi_p(w_1), \psi_p(w_3)\right\}$.

Example 3.6. Consider a semigroup $S = \{u, v, w, x, y\}$ with the above Cayley Table. Define a Pythagorean cubic set $P^c = \left\langle \left[\widetilde{\phi_p}, \widetilde{\psi_p}\right], (\phi_p, \psi_p) \right\rangle$ in S as follows.

S	$\left[\widetilde{\phi_p}(w_1), \widetilde{\psi_p}(w_1)\right]$	$(\phi_p(w_1), \psi_p(w_1))$
u	$[0.8, 0.9], [0.1, 0.3]$	$0.2, 0.7$
v	$[0.3, 0.5], [0.7, 0.9]$	$0.8, 0.3$
w	$[0.4, 0.6], [0.6, 0.7]$	$0.5, 0.4$
x	$[0.3, 0.5], [0.7, 0.9]$	$0.8, 0.3$
y	$[0.7, 0.8], [0.4, 0.5]$	$0.4, 0.6$

Thus $P^c = \left\langle \left[\widetilde{\phi_p}, \widetilde{\psi_p}\right], (\phi_p, \psi_p) \right\rangle$ is a PCBI of S.

Definition 3.7. A PCS $P^c = (\phi_{p^c}, \psi_{p^c}) = \left\langle \left[\widetilde{\phi_p}, \widetilde{\psi_p}\right], (\phi_p, \psi_p) \right\rangle$ on S is known to be a Pythagorean cubic interior ideal (P^c_{II}) of S. If for all $w_1, w_2, w_3 \in S$, and satisfy.
$\widetilde{\phi_p}\left(w_1 w_2 w_3\right) \geq \widetilde{\phi_p}(w_2)$
$\widetilde{\psi_p}\left(w_1 w_2 w_3\right) \leq \widetilde{\psi_p}(w_2)$
$\phi_p\left(w_1 w_2 w_3\right) \leq \phi_p(w_2)$
$\psi_p\left(w_1 w_2 w_3\right) \geq \psi_p(w_2)$.

Definition 3.8. For any non-empty subset N of a semigroup S is defined to be a structure $\chi_N = \left\{w_1, [\widetilde{\phi}_{\chi_N}(w_1), \widetilde{\psi}_{\chi_N}(w_1)], (\phi_{\chi_N}(w_1), \psi_{\chi_N}(w_1)) | w_1 \in S\right\}$ which is briefly denoted by $\chi_N = \left\langle [\widetilde{\phi}_{\chi_N}, \widetilde{\psi}_{\chi_N}], (\phi_{\chi_N}, \psi_{\chi_N}) \right\rangle$ where,

$$\widetilde{\phi}_{\chi_N}(w_1) = \begin{cases} \widetilde{1}\; if x \in N \\ \widetilde{0}\; otherwise \end{cases} \quad \widetilde{\psi}_{\chi_N}(w_1) = \begin{cases} \widetilde{0}\; if x \in N \\ \widetilde{1}\; otherwise \end{cases}$$

$$\phi_{\chi_N}(w_1) = \begin{cases} 0\; if x \in N \\ 1\; otherwise \end{cases} \quad \psi_{\chi_N}(w_1) = \begin{cases} 1\; if x \in N \\ 0\; otherwise \end{cases}$$

Theorem 3.1. *Let S be a semigroup. Then,*
(i) The intersection of two Pythagorean cubic sub-semigroup (PCSS) of S, is a Pythagorean cubic sub-semigroup (PCSS)of S.
(ii) The intersection of two Pythagorean cubic left(PCL)(resp. PCR)ideal of S, is PCLI(resp. PCRI) of S.

Proof. Let $P^c_1 = \left\langle \left[\widetilde{\phi}_{p_1}, \widetilde{\psi}_{p_1}\right], (\phi_{p_1}, \psi_{p_1}) \right\rangle$ and $P^c_2 = \left\langle \left[\widetilde{\phi}_{p_2}, \widetilde{\psi}_{p_2}\right], (\phi_{p_2}, \psi_{p_2}) \right\rangle$ be two Pythagorean cubic sub-semigroup of S. Let $w_1, w_2 \in S$.
Then,
(i)$\left(\widetilde{\phi}_{p_1} \cap \widetilde{\phi}_{p_2}\right)(w_1, w_2) = min\left\{\widetilde{\phi}_{p_1}(w_1, w_2), \widetilde{\phi}_{p_2}(w_1, w_2)\right\}$

$$\geq min\left\{min\left\{\widetilde{\phi}_{p_1}(w_1), \widetilde{\phi}_{p_1}(w_2)\right\}, min\left\{\widetilde{\phi}_{p_2}(w_1), \widetilde{\phi}_{p_2}(w_2)\right\}\right\}$$

$$= min\left\{min\left\{\widetilde{\phi}_{p_1}(w_1), \widetilde{\phi}_{p_2}(w_1)\right\}, min\left\{\widetilde{\phi}_{p_1}(w_2), \widetilde{\phi}_{p_2}(w_2)\right\}\right\}$$
$$= min\left\{\widetilde{\phi}_{p_1} \cap \widetilde{\phi}_{p_2}(w_1), \widetilde{\phi}_{p_1} \cap \widetilde{\phi}_{p_2}(w_2)\right\}$$

$$\left(\widetilde{\psi}_{p_1} \cup \widetilde{\psi}_{p_2}\right)(w_1, w_2) = max\left\{\widetilde{\psi}_{p_1}(w_1, w_2), \widetilde{\psi}_{p_2}(w_1, w_2)\right\}$$
$$\leq max\left\{max\left\{\widetilde{\psi}_{p_1}(w_1), \widetilde{\psi}_{p_1}(w_2)\right\}, max\left\{\widetilde{\psi}_{p_2}(w_1), \widetilde{\psi}_{p_2}(w_2)\right\}\right\}$$
$$= max\left\{max\left\{\widetilde{\psi}_{p_1}(w_1), \widetilde{\psi}_{p_2}(w_1)\right\}, max\left\{\widetilde{\psi}_{p_1}(w_2), \widetilde{\psi}_{p_2}(w_2)\right\}\right\}$$
$$= max\left\{\widetilde{\psi}_{p_1} \cup \widetilde{\psi}_{p_2}(w_1), \widetilde{\psi}_{p_1} \cup \widetilde{\psi}_{p_2}(w_2)\right\}$$

$$(\phi_{p_1} \cup \phi_{p_2})(w_1, w_2) = max\{\phi_{p_1}(w_1, w_2), \phi_{p_2}(w_1, w_2)\}$$
$$\leq max\{max\{\phi_{p_1}(w_1), \phi_{p_1}(w_2)\}, max\{\phi_{p_2}(w_1), \phi_{p_2}(w_2)\}\}$$
$$= max\{max\{\phi_{p_1}(w_1), \phi_{p_2}(w_1)\}, max\{\phi_{p_1}(w_2), \phi_{p_2}(w_2)\}\}$$
$$= max\{\phi_{p_1} \cup \phi_{p_2}(w_1), \phi_{p_1} \cup \phi_{p_2}(w_2)\}$$

$$(\psi_{p_1} \cap \psi_{p_2})(w_1, w_2) = min\{\psi_{p_1}(w_1, w_2), \psi_{p_2}(w_1, w_2)\}$$
$$\geq min\{min\{\psi_{p_1}(w_1), \psi_{p_1}(w_2)\}, min\{\psi_{p_2}(w_1), \psi_{p_2}(w_2)\}\}$$
$$= min\{min\{\psi_{p_1}(w_1), \psi_{p_2}(w_1)\}, min\{\psi_{p_1}(w_2), \psi_{p_2}(w_2)\}\}$$
$$= min\{\psi_{p_1} \cap \psi_{p_2}(w_1), \psi_{p_1} \cap \psi_{p_2}(w_2)\}$$

Therefore, $P_1^c \cap P_2^c = \left\{\left\langle\left(\widetilde{\phi}_{p_1} \cap \widetilde{\phi}_{p_2}\right), \left(\widetilde{\psi}_{p_1} \cup \widetilde{\psi}_{p_2}\right)\right\rangle, (\phi_{p_1} \cup \phi_{p_2}), (\psi_{p_1} \cap \psi_{p_2})\right\}$ PCSS of S.

(ii) $\left(\widetilde{\phi}_{p_1} \cap \widetilde{\phi}_{p_2}\right)(w_1, w_2) = min\left\{\widetilde{\phi}_{p_1}(w_1, w_2), \widetilde{\phi}_{p_2}(w_1, w_2)\right\}$

$$\geq min\left\{\widetilde{\phi}_{p_1}(w_2), \widetilde{\phi}_{p_2}(w_2)\right\}$$
$$= \left(\widetilde{\phi}_{p_1} \cap \widetilde{\phi}_{p_2}\right)(w_2)$$

$$\left(\widetilde{\psi}_{p_1} \cup \widetilde{\psi}_{p_2}\right)(w_1, w_2) = max\left\{\widetilde{\psi}_{p_1}(w_1, w_2), \widetilde{\psi}_{p_2}(w_1, w_2)\right\}$$
$$\leq max\left\{\widetilde{\psi}_{p_1}(w_2), \widetilde{\psi}_{p_2}(w_2)\right\}$$
$$= \left(\widetilde{\psi}_{p_1} \cup \widetilde{\psi}_{p_2}\right)(w_2)$$

$$(\phi_{p_1} \cup \phi_{p_2})(w_1, w_2) = max\{\phi_{p_1}(w_1, w_2), \phi_{p_2}(w_1, w_2)\}$$
$$\leq max\{\phi_{p_1}(w_2), \phi_{p_2}(w_2)\}$$
$$= (\phi_{p_1} \cup \phi_{p_2})(w_2)$$

$$(\psi_{p_1} \cap \psi_{p_2})(w_1, w_2) = min\left\{\widetilde{\psi}_{p_1}(w_1, w_2), \widetilde{\psi}_{p_2}(w_1, w_2)\right\}$$
$$\geq min\{\psi_{p_1}(w_2), \psi_{p_2}(w_2)\}$$
$$= (\psi_{p_1} \cap \psi_{p_2})(w_2).$$

Therefore, (i) and (ii) $P_1^c \cap P_2^c = \left\{\left\langle\left(\widetilde{\phi}_{p_1} \cap \widetilde{\phi}_{p_2}\right), \left(\widetilde{\psi}_{p_1} \cup \widetilde{\psi}_{p_2}\right)\right\rangle, (\phi_{p_1} \cup \phi_{p_2}), (\psi_{p_1} \cap \psi_{p_2})\right\}$ is a Pythagorean cubic left(resp. right) ideal of S.

□

Theorem 3.2. *A PCS $P^c = \left\langle\left[\widetilde{\phi}_p, \widetilde{\psi}_p\right], (\phi_p, \psi_p)\right\rangle$ of a semigroup S is a PCBI of S, iff $\left\langle\left(\phi_p^L, \phi_p^U\right), \left(\psi_p^L, \psi_p^U\right)\right\rangle$ and (ϕ_p, ψ_p) are PFI of S.*

Proof. Let $P^c = \left\langle\left[\widetilde{\phi}_p, \widetilde{\psi}_p\right], (\phi_p, \psi_p)\right\rangle$ be a PFBI(Pythagorean cubic bi-ideal) of S, for any $w_1, w_2 \in S$.

Then, we have membership

$$\left[\phi_p^L(w_1w_2), \phi_p^U(w_1w_2)\right] = \widetilde{\phi}_p(w_1w_2)$$
$$\geq min\left\{\widetilde{\phi}_p(w_1), \widetilde{\phi}_p(w_2)\right\}$$

$$= min\left\{\left[\phi_p^L(w_1), \phi_p^U(w_1)\right], \left[\phi_p^L(w_2), \phi_p^U(w_2)\right]\right\}$$
$$= min\left\{\left[\phi_p^L(w_1), \phi_p^L(w_2)\right], \left[\phi_p^U(w_1), \phi_p^U(w_2)\right]\right\}.$$

It follows that $\phi_p^L(w_1w_2) \geq min\left\{\phi_p^L(w_1), \phi_p^L(w_2)\right\}$ and $\phi_p^U(w_1w_2) \geq min\left\{\phi_p^U(w_1), \phi_p^U(w_2)\right\}$ and non-membership

$$\left[\psi_p^L(w_1w_2), \psi_p^U(w_1w_2)\right] = \widetilde{\psi}_p(w_1w_2)$$
$$\leq max\left\{\widetilde{\psi}_p(w_1), \widetilde{\psi}_p(w_2)\right\}$$
$$= max\left\{\left[\psi_p^L(w_1), \psi_p^U(w_1)\right], \left[\psi_p^L(w_2), \psi_p^U(w_2)\right]\right\}$$
$$= max\left\{\left[\psi_p^L(w_1), \psi_p^L(w_2)\right], \left[\psi_p^U(w_1), \psi_p^U(w_2)\right]\right\}.$$

It follows that $\psi_p^L(w_1w_2) \leq max\left\{\psi_p^L(w_1), \psi_p^L(w_2)\right\}$ and $\psi_p^U(w_1w_2) \leq max\left\{\psi_p^U(w_1), \phi_p^U(w_2)\right\}$
Clearly, $\phi_p(w_1w_2) \leq max\left\{\phi_p(w_1), \phi_p(w_2)\right\}$ and $\psi_p(w_1w_2) \geq min\left\{\psi_p(w_1), \psi_p(w_2)\right\}$.
Therefore, $P = \left\langle\left(\phi_p^L, \phi_p^U\right), \left(\psi_p^L, \psi_p^U\right)\right\rangle$ and (ϕ_p, ψ_p) are Pythagorean fuzzy ideal of S.
Conversely, suppose that $\left(\left[\phi_p^L, \phi_p^U\right], \left[\psi_p^L, \psi_p^U\right]\right)$ and (ϕ_p, ψ_p) are Pythagorean fuzzy ideal of S, let $w_1, w_2 \in S$.

$$\widetilde{\phi}_p(w_1w_2) = \left[\phi_p^L(w_1w_2), \phi_p^U(w_1w_2)\right]$$
$$\geq \left[min\left\{\phi_p^L(w_1), \phi_p^L(w_2)\right\}, min\left\{\phi_p^U(w_1), \phi_p^U(w_2)\right\}\right]$$
$$= min\left\{\left[\phi_p^L(w_1), \phi_p^U(w_1)\right], \left[\phi_p^L(w_2), \phi_p^U(w_2)\right]\right\}$$
$$= min\left\{\widetilde{\phi}_p(w_1), \widetilde{\phi}_p(w_2)\right\}$$

$$\widetilde{\psi}_p(w_1w_2) = \left[\psi_p^L(w_1w_2), \psi_p^U(w_1w_2)\right]$$
$$\leq \left[max\left\{\psi_p^L(w_1), \psi_p^L(w_2)\right\}, max\left\{\psi_p^U(w_1), \psi_p^U(w_2)\right\}\right]$$
$$= max\left\{\left[\psi_p^L(w_1), \psi_p^U(w_1)\right], \left[\psi_p^L(w_2), \psi_p^U(w_2)\right]\right\}$$
$$= max\left\{\widetilde{\psi}_p(w_1), \widetilde{\psi}_p(w_2)\right\}.$$

Clearly, $\phi_p(w_1w_2) \leq max\left\{\phi_p(w_1), \phi_p(w_2)\right\}$ and $\psi_p(w_1w_2) \geq min\left\{\psi_p(w_1), \psi_p(w_2)\right\}$
$P^c = \left\langle\left[\widetilde{\phi}_p, \widetilde{\psi}_p\right], (\phi_p, \psi_p)\right\rangle$ is a Pythagorean cubic sub-semigroup of S.

$$\widetilde{\phi}_p(w_1w_2w_3) = \left[\phi_p^L(w_1w_2w_3), \phi_p^U(w_1w_2w_3)\right]$$
$$\geq \left[min\left\{\phi_p^L(w_1), \phi_p^L(w_3)\right\}, min\left\{\phi_p^U(w_1), \phi_p^U(w_3)\right\}\right]$$
$$= min\left\{\left[\phi_p^L(w_1), \phi_p^U(w_1)\right], \left[\phi_p^L(w_3), \phi_p^U(w_3)\right]\right\}$$
$$= min\left\{\widetilde{\phi}_p(w_1), \widetilde{\phi}_p(w_3)\right\}$$

$$\widetilde{\psi}_p(w_1w_2w_3) = \left[\psi_p^L(w_1w_2w_3), \psi_p^U(w_1w_2w_3)\right]$$
$$\leq \left[max\left\{\psi_p^L(w_1), \psi_p^L(w_3)\right\}, max\left\{\psi_p^U(w_1), \psi_p^U(w_3)\right\}\right]$$
$$= max\left\{\left[\psi_p^L(w_1), \psi_p^U(w_1)\right], \left[\psi_p^L(w_3), \psi_p^U(w_3)\right]\right\}$$
$$= max\left\{\widetilde{\psi}_p(w_1), \widetilde{\psi}_p(w_3)\right\}.$$

Clearly, $\phi_p(w_1w_2w_3) \leq max\left\{\phi_p(w_1), \phi_p(w_3)\right\}$ and $\psi_p(w_1w_2w_3) \geq min\left\{\psi_p(w_1), \psi_p(w_3)\right\}$
$P^c = \left\langle\left[\widetilde{\phi}_p, \widetilde{\psi}_p\right], (\phi_p, \psi_p)\right\rangle$ is a Pythagorean cubic bi-ideal of S. □

Theorem 3.3. *If $\{P_i\}_{i\in I}$ is a family of Pythagorean cubic bi-ideal of a semigroup S. Then $\cap P_i$ is a Pythagorean cubic bi-ideal of S. Where $\cap P_i = \left(\left(\cap\widetilde{\phi}_{p_i}, \cup\widetilde{\psi}_{p_i}\right), (\cup\phi_{p_i}, \cap\psi_{p_i})\right)$.*
$\cap\left(\widetilde{\phi}_{p_i}\right) = inf\left\{\left(\widetilde{\phi}_{p_i}\right)(w_1)/i \in I, w_1 \in S\right\}$,
$\cup\left(\widetilde{\psi}_{p_i}\right) = sup\left\{\left(\widetilde{\psi}_{p_i}\right)(w_1)/i \in I, w_1 \in S\right\}$,
$\cup(\phi_{p_i}) = sup\left\{(\phi_{p_i})(w_1)/i \in I, w_1 \in S\right\}$,
$\cap(\psi_{p_i}) = inf\left\{(\psi_{p_i})(w_1)/i \in I, w_1 \in S\right\}$ and $i \in I$ is any index set.

Proof. Since $P_i = \left\langle \left[\widetilde{\phi}_{p_i}, \widetilde{\psi}_{p_i}\right], (\phi_{p_i}, \psi_{p_i})\, | i \in I \right\rangle$ is a family of Pythagorean cubic bi-ideals of S. Let $w_1, w_2, w_3 \in S$.

$$\begin{aligned}
\cap\widetilde{\phi}_{p_i}(w_1, w_2) &= inf\left\{\widetilde{\phi}_{p_i}(w_1, w_2)/i \in I, w_1, w_2 \in S\right\} \\
&\geq inf\left\{min\left\{\widetilde{\phi}_{p_i}(w_1), \widetilde{\phi}_{p_i}(w_2)\right\}\right\} \\
&= min\left\{inf\left(\widetilde{\phi}_{p_i}(w_1)\right), inf\left(\widetilde{\phi}_{p_i}(w_2)\right)\right\} \\
&= min\left\{\cap\widetilde{\phi}_{p_i}(w_1), \cap\widetilde{\phi}_{p_i}(w_2)\right\}
\end{aligned}$$

$$\begin{aligned}
\cup\widetilde{\psi}_{p_i}(w_1 w_2) &= sup\left\{\widetilde{\psi}_{p_i}(w_1 w_2)/i \in I, w_1, w_2 \in S\right\} \\
&\leq sup\left\{max\left\{\widetilde{\psi}_{p_i}(w_1), \widetilde{\psi}_{p_i}(w_2)\right\}\right\} \\
&= max\left\{sup\left(\widetilde{\psi}_{p_i}(w_1)\right), sup\left(\widetilde{\psi}_{p_i}(w_2)\right)\right\} \\
&= max\left\{\cup\widetilde{\psi}_{p_i}(w_1), \cup\widetilde{\psi}_{p_i}(w_2)\right\}
\end{aligned}$$

$$\begin{aligned}
\cup\phi_{p_i}(w_1 w_2) &= sup\left\{\phi_{p_i}(w_1 w_2)/i \in I, w_1, w_2 \in S\right\} \\
&\leq sup\left\{max\left\{\phi_{p_i}(w_1), \phi_{p_i}(w_2)\right\}\right\} \\
&= max\left\{sup\left(\phi_{p_i}(w_1)\right), sup\left(\phi_{p_i}(w_2)\right)\right\} \\
&= max\left\{\cup\phi_{p_i}(w_1), \cup\phi_{p_i}(w_2)\right\}
\end{aligned}$$

$$\begin{aligned}
\cap\psi_{p_i}(w_1 w_2) &= inf\left\{\psi_{p_i}(w_1 w_2)/i \in I, w_1, w_2 \in S\right\} \\
&\geq inf\left\{min\left\{\psi_{p_i}(w_1), \psi_{p_i}(w_2)\right\}\right\} \\
&= min\left\{inf\left(\psi_{p_i}(w_1)\right), inf\left(\psi_{p_i}(w_2)\right)\right\} \\
&= min\left\{\cap\psi_{p_i}(w_1), \cap\psi_{p_i}(w_2)\right\}.
\end{aligned}$$

Hence, $\cap P_i^c = \left(\left(\cap\widetilde{\phi}_{p_i}, \cup\widetilde{\psi}_{p_i}\right), (\cup\phi_{p_i}, \cap\psi_{p_i})\right)$ is a Pythagorean cubic sub-semigorup of S.

$$\begin{aligned}
\cap\widetilde{\phi}_{p_i}(w_1 w_2 w_3) &= inf\left\{\widetilde{\phi}_{p_i}(w_1, w_2 w_3)/i \in I, w_1, w_2, w_3 \in S\right\} \\
&\geq inf\left\{min\left\{\widetilde{\phi}_{p_i}(w_1), \widetilde{\phi}_{p_i}(w_3)\right\}\right\} \\
&= min\left\{inf\left(\widetilde{\phi}_{p_i}(w_1)\right), inf\left(\widetilde{\phi}_{p_i}(w_3)\right)\right\} \\
&= min\left\{\cap\widetilde{\phi}_{p_i}(w_1), \cap\widetilde{\phi}_{p_i}(w_3)\right\}
\end{aligned}$$

$$\begin{aligned}
\cup\widetilde{\psi}_{p_i}(w_1 w_2 w_3) &= sup\left\{\widetilde{\psi}_{p_i}(w_1 w_2 w_3)/i \in I, w_1, w_2, w_3 \in S\right\} \\
&\leq sup\left\{max\left\{\widetilde{\psi}_{p_i}(w_1), \widetilde{\psi}_{p_i}(w_3)\right\}\right\} \\
&= max\left\{sup\left(\widetilde{\psi}_{p_i}(w_1)\right), sup\left(\widetilde{\psi}_{p_i}(w_3)\right)\right\} \\
&= max\left\{\cup\widetilde{\psi}_{p_i}(w_1), \cup\widetilde{\psi}_{p_i}(w_3)\right\}
\end{aligned}$$

$$\begin{aligned}
\cup\phi_{p_i}(w_1 w_2 w_3) &= sup\left\{\phi_{p_i}(w_1 w_2 w_3)/i \in I, w_1, w_2, w_3 \in S\right\} \\
&\leq sup\left\{max\left\{\phi_{p_i}(w_1), \phi_{p_i}(w_3)\right\}\right\} \\
&= max\left\{sup\left(\phi_{p_i}(w_1)\right), sup\left(\phi_{p_i}(w_3)\right)\right\} \\
&= max\left\{\cup\phi_{p_i}(w_1), \cup\phi_{p_i}(w_3)\right\}
\end{aligned}$$

$$\begin{aligned}
\cap\psi_{p_i}(w_1 w_2 w_3) &= inf\left\{\psi_{p_i}(w_1 w_2 w_3)/i \in I, w_1, w_2, w_3 \in S\right\} \\
&\geq inf\left\{min\left\{\psi_{p_i}(w_1), \psi_{p_i}(w_3)\right\}\right\} \\
&= min\left\{inf\left(\psi_{p_i}(w_1)\right), inf\left(\psi_{p_i}(w_3)\right)\right\} \\
&= min\left\{\cap\psi_{p_i}(w_1), \cap\psi_{p_i}(w_3)\right\}.
\end{aligned}$$

Hence, $\cap P_i^c = \left(\left(\cap\widetilde{\phi}_{p_i}, \cup\widetilde{\psi}_{p_i}\right), (\cup\phi_{p_i}, \cap\psi_{p_i})\right)$ is a PCBI of S. □

Theorem 3.4. *Let N be any non-empty subset of a semigroup S. Then N is a bi-ideal of S, iff the characteristic Pythagorean cubic set $\chi_N = \left\langle \left[\widetilde{\phi}_{p\chi_N}, \widetilde{\psi}_{p\chi_N}\right], (\phi_{p\chi_N}, \psi_{p\chi_N})\right\rangle$ is PCBI of S.*

Proof. Assume that N is a bi-ideal of S. Let $w_1, w_2, w_3 \in S$. Suppose that $\widetilde{\phi}_{p\chi_N}(w_1w_2) < min\left\{\widetilde{\phi}_{p\chi_N}(w_1), \widetilde{\phi}_{p\chi_N}(w_2)\right\}$ and $\widetilde{\psi}_{p\chi_N}(w_1w_2) > max\left\{\widetilde{\psi}_{p\chi_N}(w_1), \widetilde{\psi}_{p\chi_N}(w_2)\right\}$ it follows that $\widetilde{\phi}_{p\chi_N}(w_1w_2) = 0, min\left\{\widetilde{\phi}_{p\chi_N}(w_1), \widetilde{\phi}_{p\chi_N}(w_2)\right\} = 1$, $\widetilde{\psi}_{p\chi_N}(w_1w_2) = 1$, $max\left\{\widetilde{\psi}_{p\chi_N}(w_1), \widetilde{\psi}_{p\chi_N}(w_2)\right\} = 0$, $\phi_{p\chi_N}(w_1w_2) > max\left\{\phi_{p\chi_N}(w_1), \phi_{p\chi_N}(w_2)\right\}$ and $\psi_{p\chi_N}(w_1w_2) < min\left\{\psi_{p\chi_N}(w_1), \psi_{p\chi_N}(w_2)\right\}$ it follows that $\phi_{p\chi_N}(w_1w_2) = 1$, $max\left\{\phi_{p\chi_N}(w_1), \phi_{p\chi_N}(w_2)\right\} = 0$, $\psi_{p\chi_N}(w_1w_2) = 0, min\left\{\psi_{p\chi_N}(w_1), \psi_{p\chi_N}(w_2)\right\} = 1$.
This implies that $w_1, w_2 \in N$ by $w_1, w_2 \notin N$ a contradiction to N. So $\widetilde{\phi}_{p\chi_N}(w_1w_2) \geq min\left\{\widetilde{\phi}_{p\chi_N}(w_1), \widetilde{\phi}_{p\chi_N}(w_2)\right\}$, $\widetilde{\psi}_{p\chi_N}(w_1w_2) \leq max\left\{\widetilde{\psi}_{p\chi_N}(w_1), \widetilde{\psi}_{p\chi_N}(w_2)\right\}$, $\phi_{p\chi_N}(w_1w_2) \leq max\left\{\phi_{p\chi_N}(w_1), \phi_{p\chi_N}(w_2)\right\}$ and $\psi_{p\chi_N}(w_1w_2) \geq min\left\{\psi_{p\chi_N}(w_1), \psi_{p\chi_N}(w_2)\right\}$.
Suppose that $\widetilde{\phi}_{p\chi_N}(w_1w_2w_3) < min\left\{\widetilde{\phi}_{p\chi_N}(w_1), \widetilde{\phi}_{p\chi_N}(w_3)\right\}$ and
$\widetilde{\psi}_{p\chi_N}(w_1w_2w_3) > max\left\{\widetilde{\psi}_{p\chi_N}(w_1), \widetilde{\psi}_{p\chi_N}(w_3)\right\}$ it follows that
$\widetilde{\phi}_{p\chi_N}(w_1w_2w_3) = 0, min\left\{\widetilde{\phi}_{p\chi_N}(w_1), \widetilde{\phi}_{p\chi_N}(w_3)\right\} = 1$,
$\widetilde{\psi}_{p\chi_N}(w_1w_2w_3) = 1, max\left\{\widetilde{\psi}_{p\chi_N}(w_1), \widetilde{\psi}_{p\chi_N}(w_3)\right\} = 0$,
$\phi_{p\chi_N}(w_1w_2w_3) > max\left\{\phi_{p\chi_N}(w_1), \phi_{p\chi_N}(w_3)\right\}$
and $\psi_{p\chi_N}(w_1w_2w_3) < min\left\{\psi_{p\chi_N}(w_1), \psi_{p\chi_N}(w_3)\right\}$ it follows that
$\phi_{p\chi_N}(w_1w_2w_3) = 1, max\left\{\phi_{p\chi_N}(w_1), \phi_{p\chi_N}(w_3)\right\} = 0$,
$\psi_{p\chi_N}(w_1w_2w_3) = 0, min\left\{\psi_{p\chi_N}(w_1), \psi_{p\chi_N}(w_3)\right\} = 1$.
This implies that $w_1, w_2, w_3 \in N$ by $w_1, w_2, w_3 \notin N$ a contradiction to N.
So $\widetilde{\phi}_{p\chi_N}(w_1w_2w_3) \geq min\left\{\widetilde{\phi}_{p\chi_N}(w_1), \widetilde{\phi}_{p\chi_N}(w_3)\right\}$,
$\widetilde{\psi}_{p\chi_N}(w_1w_2w_3) \leq max\left\{\widetilde{\psi}_{p\chi_N}(w_1), \widetilde{\psi}_{p\chi_N}(w_3)\right\}$,
$\phi_{p\chi_N}(w_1w_2w_3) \leq max\left\{\phi_{p\chi_N}(w_1), \phi_{p\chi_N}(w_3)\right\}$
and $\psi_{p\chi_N}(w_1w_2w_3) \geq min\left\{\psi_{p\chi_N}(w_1), \psi_{p\chi_N}(w_3)\right\}$.
This shows that χ_N is a Pythagorean cubic bi-ideal of S.
Conversely, $\chi_N = \left\langle \left[\widetilde{\phi}_{p\chi_N}, \widetilde{\psi}_{p\chi_N}\right], (\phi_{p\chi_N}, \psi_{p\chi_N})\right\rangle$ is PCBI of S for any subset N of S.
Let $w_1, w_2 \in N$ then $\widetilde{\phi}_{p\chi_N}(w_1) = \widetilde{\phi}_{p\chi_N}(w_2) = \widetilde{1}$, $\widetilde{\psi}_{p\chi_N}(w_1) = \widetilde{\psi}_{p\chi_N}(w_2) = \widetilde{0}$ and $\phi_{p\chi_N}(w_1) = \phi_{p\chi_N}(w_2) = \widetilde{0}$ and $\psi_{p\chi_N}(w_1) = \psi_{p\chi_N}(w_2) = \widetilde{1}$ since χ_N is a PCBI of S.
$\widetilde{\phi}_{p\chi_N}(w_1w_2) \geq min\left\{\widetilde{\phi}_{p\chi_N}(w_1), \widetilde{\phi}_{p\chi_N}(w_2)\right\} \geq min\{\widetilde{1}, \widetilde{1}\} = \widetilde{1}$,
$\widetilde{\psi}_{p\chi_N}(w_1w_2) \leq max\left\{\widetilde{\psi}_{p\chi_N}(w_1), \widetilde{\psi}_{p\chi_N}(w_2)\right\} \leq max\{\widetilde{0}, \widetilde{0}\} = \widetilde{0}$,
$\phi_{p\chi_N}(w_1w_2) \leq max\left\{\phi_{p\chi_N}(w_1), \phi_{p\chi_N}(w_2)\right\} \leq max\{\widetilde{0}, \widetilde{0}\} = \widetilde{0}$
and $\psi_{p\chi_N}(w_1w_2) \geq min\left\{\psi_{p\chi_N}(w_1), \psi_{p\chi_N}(w_2)\right\} \geq min\{\widetilde{1}, \widetilde{1}\} = \widetilde{1}$.
This implies that $w_1w_2 \in N$.
Let $w_1, w_2w_3 \in N$ then $\widetilde{\phi}_{p\chi_N}(w_1) = \widetilde{\phi}_{p\chi_N}(w_2) = \widetilde{\phi}_{p\chi_N}(w_3) = \widetilde{1}$, $\widetilde{\psi}_{p\chi_N}(w_1) = \widetilde{\psi}_{p\chi_N}(w_2) = \widetilde{\psi}_{p\chi_N}(w_3) = \widetilde{0}$ and $\phi_{p\chi_N}(w_1) = \phi_{p\chi_N}(w_2) = \phi_{p\chi_N}(w_3) = \widetilde{0}$ and $\psi_{p\chi_N}(w_1) = \psi_{p\chi_N}(w_2) = \psi_{p\chi_N}(w_3) = \widetilde{1}$ since χ_N is a PCBI of S. $\widetilde{\phi}_{p\chi_N}(w_1w_2w_3) \geq min\left\{\widetilde{\phi}_{p\chi_N}(w_1), \widetilde{\phi}_{p\chi_N}(w_3)\right\} \geq min\{\widetilde{1}, \widetilde{1}\} = \widetilde{1}$, $\widetilde{\psi}_{p\chi_N}(w_1w_2w_3) \leq max\left\{\widetilde{\psi}_{p\chi_N}(w_1), \widetilde{\psi}_{p\chi_N}(w_3)\right\} \leq$

$max\{\widetilde{0},\widetilde{0}\} = \widetilde{0}$,
$\phi_{p\chi_N}(w_1w_2w_3) \leq max\{\phi_{p\chi_N}(w_1), \phi_{p\chi_N}(w_3)\} \leq max\{\widetilde{0},\widetilde{0}\} = \widetilde{0}$ and $\psi_{p\chi_N}(w_1w_2w_3) \geq min\{\psi_{p\chi_N}(w_1), \psi_{p\chi_N}(w_3)\} \geq min\{\widetilde{1},\widetilde{1}\} = \widetilde{1}$. Which implies that $w_1w_2 \in N$.
Hence N is a bi- ideal of S. □

Theorem 3.5. *If $\{P_i\}_{i\in I}$ is a family of Pythagorean cubic interior ideal of a semigroup S. Then $\cap P_i$ is a Pythagorean cubic interior ideal of S. Where* $\cap P_i = \left(\left(\cap\widetilde{\phi}_{p_i}, \cup\widetilde{\psi}_{p_i}\right), (\cup\phi_{p_i}, \cap\psi_{p_i})\right)$.
$\cap\left(\widetilde{\phi}_{p_i}\right) = inf\left\{\left(\widetilde{\phi}_{p_i}\right)(w_1)/i \in I, w_1 \in S\right\}$,
$\cup\left(\widetilde{\psi}_{p_i}\right) = sup\left\{\left(\widetilde{\psi}_{p_i}\right)(w_1)/i \in I, w_1 \in S\right\}$,
$\cup(\phi_{p_i}) = sup\{(\phi_{p_i})(w_1)/i \in I, w_1 \in S\}$,
$\cap(\psi_{p_i}) = inf\{(\psi_{p_i})(w_1)/i \in I, w_1 \in S\}$ *and $i \in I$ is any index set.*

Theorem 3.6. *Let N be any non-empty subset of a semigroup S. Then N is a interior ideal of S, iff the characteristic Pythagorean cubic set* $\chi_N = \left\langle\left[\widetilde{\phi}_{p\chi_N}, \widetilde{\psi}_{p\chi_N}\right], (\phi_{p\chi_N}, \psi_{p\chi_N})\right\rangle$ *is PCII of S.*

4. HOMOMORPHISM OF PYTHAGOREAN CUBIC IDEAL IN SEMIGROUP

Let R and T be two non-empty sets of semigroup S. A mapping $f : R \to T$ is called a homomorphism if $f(rt) = f(r)f(t)\ \forall r, t \in R$.

Definition 4.1. Let f be a mapping from a set R to a set T and $P^c = \left\langle\left[\widetilde{\phi}_p, \widetilde{\psi}_p\right], (\phi_p, \psi_p)\right\rangle$ be a Pythagorean cubic set R the the image of R (i.e.,) $f(P^c) = (f(\widetilde{\phi}_p), f(\widetilde{\psi}_p), f(\phi_p), f(\psi_p))$ is a Pythagorean cubic set of T is defined by

$$f(P^c)(r) = \begin{cases} f(\widetilde{\phi_P})(r) = \begin{cases} \sup\limits_{t\in f'(r)} (\widetilde{\phi_P})(t), & if\ f^{-1}(r) = 0 \\ [0,0] & otherwise \end{cases} \\ f(\widetilde{\psi_P})(r) = \begin{cases} \inf\limits_{t\in f'(r)} (\widetilde{\psi_P})(t), & if\ f^{-1}(r) = 0 \\ [1,1] & otherwise \end{cases} \\ f(\phi_P)(r) = \begin{cases} \inf\limits_{t\in f'(r)} (\phi_P)(t), & if\ f^{-1}(r) = 0 \\ [1,1] & otherwise \end{cases} \\ f(\psi_P)(r) = \begin{cases} \sup\limits_{t\in f'(r)} (\psi_P)(t), & if\ f^{-1}(r) = 0 \\ [0,0] & otherwise \end{cases} \end{cases}$$

Let f be a mapping from a set R to T and $P^c = \left\langle\left[\widetilde{\phi}_p, \widetilde{\psi}_p\right], (\phi_p, \psi_p)\right\rangle$ be a Pythagorean cubic set of T then the pre image of T (i.e.,) $f^{-1}(P^c) = \left\{(f^{-1}(\widetilde{\phi}_p), f^{-1}(\widetilde{\psi}_p)), (f^{-1}(\phi_p), f^{-1}(\psi_p))\right\}$ is an Pythagorean cubic set of R is defined as

$$f^{-1}(P^c)(r) = \begin{cases} f^{-1}(\widetilde{\phi_p})(r) = \widetilde{\phi_p}(f(r)) \\ f^{-1}(\widetilde{\psi_p})(r) = \widetilde{\psi_p}(f(r)) \\ f^{-1}(\phi_p)(r) = \phi_p(f(r)) \\ f^{-1}(\psi_p)(r) = \psi_p(f(r)) \end{cases}$$

Theorem 4.1. *Let R, T be a semigroups, $f : R \to T$ be a homomorphism of semigroups.*
(a) If $P^c = \left\langle \left[\widetilde{\phi}_p, \widetilde{\psi}_p\right], (\phi_p, \psi_p)\right\rangle$ is a Pythagorean cubic sub-semigroup of T the the preimage$f^{-1}(P^c) = (f^{-1}(\widetilde{\phi}_p), f^{-1}(\widetilde{\psi}_p), f^{-1}(\phi_p), f^{-1}(\psi_p))$ is a Pythagorean cubic sub-semigroup of R.
(b) If $P^c = \left\langle \left[\widetilde{\phi}_p, \widetilde{\psi}_p\right], (\phi_p, \psi_p)\right\rangle$ is a Pythagorean cubic left (resp.right) ideal of T the the preimage$f^{-1}(P^c) = (f^{-1}(\widetilde{\phi}_p), f^{-1}(\widetilde{\psi}_p), f^{-1}(\phi_p), f^{-1}(\psi_p))$ is a Pythagorean cubic left ideal (resp. right ideal) of R.

Proof. Assume that $P^c = \left\langle \left[\widetilde{\phi}_p, \widetilde{\psi}_p\right], (\phi_p, \psi_p)\right\rangle$ is a Pythagorean cubic sub-semigroup of T and $r, t \in R$. Then

(a)(i) $$\begin{aligned} f^{-1}(\widetilde{\phi}_p)(rt) &= \widetilde{\phi}_p(f(rt)) \\ &= \widetilde{\phi}_p(f(r)f(t)) \\ &\geq min\left\{\widetilde{\phi}_p(f(r)), \widetilde{\phi}_p(f(t))\right\} \\ &= min\left\{f^{-1}(\widetilde{\phi}_p)(r), f^{-1}(\widetilde{\phi}_p)(f(t))\right\} \end{aligned}$$

(ii) $$\begin{aligned} f^{-1}(\widetilde{\psi}_p)(rt) &= \widetilde{\psi}_p(f(rt)) \\ &= \widetilde{\psi}_p(f(r)f(t)) \\ &\leq max\left\{\widetilde{\psi}_p(f(r)), \widetilde{\psi}_p(f(t))\right\} \\ &= max\left\{f^{-1}(\widetilde{\psi}_p)(r), f^{-1}(\widetilde{\psi}_p)(f(t))\right\} \end{aligned}$$

(iii) $$\begin{aligned} f^{-1}(\phi_p)(rt) &= \phi_p(f(rt)) \\ &= \phi_p(f(r)f(t)) \\ &\leq max\left\{\phi_p(f(r)), \phi_p(f(t))\right\} \\ &= max\left\{f^{-1}(\phi_p)(r), f^{-1}(\phi_p)(f(t))\right\} \end{aligned}$$

(iv) $$\begin{aligned} f^{-1}(\psi_p)(rt) &= \psi_p(f(rt)) \\ &= \psi_p(f(r)f(t)) \\ &\geq min\left\{\psi_p(f(r)), \psi_p(f(t))\right\} \\ &= min\left\{f^{-1}(\psi_p)(r), f^{-1}(\psi_p)(f(t))\right\}. \end{aligned}$$

Hence, $f^{-1}(P^c) = (f^{-1}(\widetilde{\phi}_p), f^{-1}(\widetilde{\psi}_p), f^{-1}(\phi_p), f^{-1}(\psi_p))$ is a Pythagorean cubic sub-semigroup of R.

(b)(i) $$\begin{aligned} f^{-1}(\widetilde{\phi}_p)(rt) &= \widetilde{\phi}_p(f(rt)) \\ &= \widetilde{\phi}_p(f(r)f(t)) \\ &\geq \widetilde{\phi}_p(f(t)) \\ &= f^{-1}(\widetilde{\phi}_p)(f(t)) \end{aligned}$$

(ii) $$\begin{aligned} f^{-1}(\widetilde{\psi}_p)(rt) &= \widetilde{\psi}_p(f(rt)) \\ &= \widetilde{\psi}_p(f(r)f(t)) \\ &\leq \widetilde{\psi}_p(f(t)) \\ &= f^{-1}(\widetilde{\psi}_p)(f(t)) \end{aligned}$$

(iii) $$\begin{aligned} f^{-1}(\phi_p)(rt) &= \phi_p(f(rt)) \\ &= \phi_p(f(r)f(t)) \\ &\leq \phi_p(f(t)) \\ &= f^{-1}(\phi_p)(f(t)) \end{aligned}$$

(iv) $$\begin{aligned} f^{-1}(\psi_p)(rt) &= \psi_p(f(rt)) \\ &= \psi_p(f(r)f(t)) \end{aligned}$$

$\geq \psi_p(f(t))$
$= f^{-1}(\psi_p)(f(t))$.

Hence, $f^{-1}(P^c) = (f^{-1}(\widetilde{\phi}_p), f^{-1}(\widetilde{\psi}_p), f^{-1}(\phi_p), f^{-1}(\psi_p))$ is a Pythagorean cubic left(resp.right) ideal of R. □

Theorem 4.2. *Let R, T be a semigroups, $f : R \to T$ be a homomorphism of semigroups. If $P^c = \left\langle \left[\widetilde{\phi}_p, \widetilde{\psi}_p\right], (\phi_p, \psi_p) \right\rangle$ is a Pythagorean cubic bi-ideal of T the the preimage$f^{-1}(P^c) = (f^{-1}(\widetilde{\phi}_p), f^{-1}(\widetilde{\psi}_p), f^{-1}(\phi_p), f^{-1}(\psi_p))$ is a Pythagorean cubic bi-ideal of R.*

Proof. Assume that $P^c = \left\langle \left[\widetilde{\phi}_p, \widetilde{\psi}_p\right], (\phi_p, \psi_p) \right\rangle$ is a Pythagorean cubic sub-semigroup of T and $a, r, t \in R$. Then

(a)(i) $f^{-1}(\widetilde{\phi}_p)(rat) = \widetilde{\phi}_p(f(rat))$
$= \widetilde{\phi}_p(f(r)f(a)f(t))$
$\geq min\left\{\widetilde{\phi}_p(f(r)), \widetilde{\phi}_p(f(t))\right\}$
$= min\left\{f^{-1}(\widetilde{\phi}_p)(r), f^{-1}(\widetilde{\phi}_p)(f(t))\right\}$

(ii) $f^{-1}(\widetilde{\psi}_p)(rat) = \widetilde{\psi}_p(f(rat))$
$= \widetilde{\psi}_p(f(r)f(a)f(t))$
$\leq max\left\{\widetilde{\psi}_p(f(r)), \widetilde{\psi}_p(f(t))\right\}$
$= max\left\{f^{-1}(\widetilde{\psi}_p)(r), f^{-1}(\widetilde{\psi}_p)(f(t))\right\}$

(iii) $f^{-1}(\phi_p)(rat) = \phi_p(f(rat))$
$= \phi_p(f(r)f(a)f(t))$
$\leq max\{\phi_p(f(r)), \phi_p(f(t))\}$
$= max\{f^{-1}(\phi_p)(r), f^{-1}(\phi_p)(f(t))\}$

(iv) $f^{-1}(\psi_p)(rat) = \psi_p(f(rat))$
$= \psi_p(f(r)f(a)f(t))$
$\geq min\{\psi_p(f(r)), \psi_p(f(t))\}$
$= min\{f^{-1}(\psi_p)(r), f^{-1}(\psi_p)(f(t))\}$.

Hence $f^{-1}(P^c) = (f^{-1}(\widetilde{\phi}_p), f^{-1}(\widetilde{\psi}_p), f^{-1}(\phi_p), f^{-1}(\psi_p))$ is a Pythagorean cubic bi-ideal of R. □

Theorem 4.3. *Let R, T be a semigroups, $f : R \to T$ be a homomorphism of semigroups. If $P^c = \left\langle \left[\widetilde{\phi}_p, \widetilde{\psi}_p\right], (\phi_p, \psi_p) \right\rangle$ is a Pythagorean cubic interior ideal of T the preimage $f^{-1}(P^c) = (f^{-1}(\widetilde{\phi}_p), f^{-1}(\widetilde{\psi}_p), f^{-1}(\phi_p), f^{-1}(\psi_p))$ is a Pythagorean cubic interior ideal of R.*

5. CONCLUSIONS

In this paper we have obtained the union, intersection of PCI. The properties of Pythagorean cubic left(right) ideals, bi-ideals, interior ideals and sub-semigroup of semigroup.

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