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A NOTE ON A COUPLED SYSTEM OF CAPUTO-FABRIZIO FRACTIONAL DIFFERENTIAL INCLUSIONS

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ABSTRACT. A coupled system of Caputo-Fabrizio fractional differential inclusions is studied and the existence of solutions is obtained when the set-valued maps that define the problem have nonconvex values, but there are Lipschitz in the state variables.

1. INTRODUCTION

This note is concerned with the following Cauchy problem associated to a coupled system of Caputo-Fabrizio fractional differential inclusions

$$\begin{cases} D_{CF}^{\sigma_1}y(t) \in G(t, y(t), z(t)) & a.e.\ t \in [0,T],\ y(0) = y_0,\ y'(0) = y_1, \\ D_{CF}^{\sigma_2}z(t) \in H(t, y(t), z(t)) & a.e.\ t \in [0,T],\ z(0) = z_0,\ z'(0) = z_1, \end{cases} \tag{1.1}$$

where $\sigma_1, \sigma_2 \in (1,2)$, $G : [0,T] \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$, $H : [0,T] \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ are set-valued maps, $y_0, z_0, y_1, z_1 \in \mathbf{R}$, D_{CF}^{σ} denotes Caputo-Fabrizio's fractional derivative of order σ and $\mathcal{P}(\mathbf{R})$ is the family of all nonempty subsets of $\mathbf{R}$.

A rather new fractional order derivative was introduced by Caputo and Fabrizio in [8]. Even if it has a regular kernel this fractional derivative turned out to be very useful in applied problems [4, 5, 18] etc.. Several qualitative results for fractional differential equations and inclusions defined by Caputo-Fabrizio fractional operator may be found in [10, 11, 17, 19, 20] etc.. For more general considerations concerning the analysis of fractional differential equations we refer to [7, 13, 15].

In the last years an increasing number of papers devoted to the study of the existence of solutions for different classes of coupled systems of fractional differential equations with several boundary conditions may be found in the literature [1, 2, 3, 16] etc.. It is worth to mention that the first paper which deals with a coupled system of Caputo-Fabrizio fractional differential equations is [2]. All the results in the papers quoted above are obtained by using several fixed point techniques.

Our goal is to study problem (1.1) in the situation when the set-valued maps are not convex valued and to deduce an existence result for this problem using Filippov's technique [14]. Our main hypothsis is that the set-valued maps G and H are Lipschitz in the state variables.

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On one hand, our main result extends Theorem 3.2 in [10] obtained for a simple fractional differential inclusion of Caputo-Fabrizio type to a coupled system of such fractional differential inclusions and, on the other hand, the present paper extends the study in [2] from fractional differential equations framework to fractional differential inclusions framework.

We underline that from our main result, in a particular case, we obtain, as a corollary, an existence result which has a less complicated statement. Such a consequence may also be obtained using Covitz-Nadler set-valued contraction principle but this approach is weaker than Filippov's type approach: stronger hypotheses and without a priori bounds for solutions (e.g., [9]).

We mention that these type of results may be found in the literature [9, 12], but their account for coupled systems of Caputo-Fabrizio fractional differential inclusions is new.

2. Preliminaries

Let denote by I the interval $[0, T]$, $T > 0$ and, as usual, we denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I, \mathbf{R})$ the Banach space of all integrable functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_1 = \int_0^T |x(t)|dt$. The Pompeiu-Hausdorff distance of the closed subsets $A, B \subset \mathbf{R}$ is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$, where $d^*(A, B) = \sup\{d(a, B); a \in A\}$ and $d(x, B) = \inf_{y\in B} d(x, y)$.

The next definitions were introduced in [8].

Definition 2.1. a) The Caputo-Fabrizio integral of order $\sigma \in (0, 1)$ of a function $h \in AC_{loc}([0, \infty), \mathbf{R})$ (which means that $h'(.)$ is integrable on $[0, T]$ for any $T > 0$) is defined by

$$I^\sigma_{CF}h(t) = (1 - \sigma)h(t) + \sigma \int_0^t h(s)ds.$$

b) The Caputo-Fabrizio fractional derivative of order $\sigma \in (0, 1)$ of h is defined for $t \geq 0$ by

$$D^\sigma_{CF}f(t) = \frac{1}{1 - \sigma}\int_0^t e^{-\frac{\sigma}{1-\sigma}(t-s)}h'(s)ds.$$

c) The Caputo-Fabrizio fractional derivative of order $\alpha = \sigma + n$, $\sigma \in (0, 1)$ $n \in \mathbf{N}$ of h is defined by

$$D^\alpha_{CF}h(t) = D^\sigma_{CF}(D^n_{CF}h(t)).$$

In particular, if $\alpha = \sigma + 1$, $\sigma \in (0, 1)$ $D^\alpha_{CF}h(t) = \frac{1}{1-\sigma}\int_0^t e^{-\frac{\sigma}{1-\sigma}(t-s)}h''(s)ds.$

Lemma 2.1. *([19]) For $\alpha = \sigma + 1$, $\sigma \in (0, 1)$ and $h(.) \in L^1(I, \mathbf{R})$ the initial value problem*

$$D^\sigma_{CF}y(t) = h(t), \quad y(0) = y_0,\ y'(0) = y_1,$$

has a unique solution given by

$$y(t) = y_0 + y_1t + (1 - \sigma)\int_0^t h(s)ds + \sigma\int_0^t (t - s)h(s)ds. \tag{2.1}$$

Remark. *If we define $K(t, s) = (1 - \sigma) + \sigma(t - s)$ then the solution in (2.1) may be written as $y(t) = y_0 + y_1t + \int_0^t K(t, s)h(s)ds$. Moreover, for any $s, t \in I$, $|K(t, s)| \leq (1 - \sigma) + \sigma T =: k$.*

Definition 2.2. $(y(.), z(.)) \in AC(I, \mathbf{R}) \times AC(I, \mathbf{R})$ is said to be a solution of problem (1.1) if there exist $(g(.), h(.)) \in L^1(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ such that $g(t) \in G(t, y(t), z(t))$ *a.e.*

(I), $h(t) \in H(t, y(t), z(t))$ *a.e.* (I) and $D_{CF}^{\sigma_1} y(t) = g(t)$, $t \in I$, $y(0) = y_0, y'(0) = y_1$ $D_{CF}^{\sigma_2} z(t) = h(t)$, $t \in I$, $z(0) = z_0, z'(0) = z_1$.

In the proof of our main result we need the following classical selection result for set-valued maps (e.g., [6]).

Lemma 2.2. *Let B be the closed unit ball in $\mathbf{R}$, $F : I \to \mathcal{P}(\mathbf{R})$ is a set-valued map with nonempty closed values and $a : I \to \mathbf{R}, b : I \to \mathbf{R}_+$ are measurable functions. If $F(t) \cap (a(t) + b(t)B) \neq \emptyset$ a.e. (I), then the set-valued map $t \to F(t) \cap (a(t) + b(t)B)$ admits a measurable selection.*

3. MAIN RESULT

In what follows we are working under the next assumptions.

Hypothesis 3.1. i) $G : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ and $H : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ have nonempty closed values and the set-valued maps $t \to G(t, u, v)$, $t \to H(t, u, v)$ are measurable for any $u, v \in \mathbf{R}$.

ii) There exist $p(.), q(.) \in L^1(I, (0, \infty))$ such that, for almost all $t \in I$, $G(t, ., .)$ is $p(t)$-Lipschitz and $H(t, ., .)$ is $q(t)$-Lipschitz in the sense that

$$d_H(G(t, x_1, y_1), G(t, x_2, y_2)) \leq p(t)(|x_1 - x_2| + |y_1 - y_2|) \quad \forall\, x_1, x_2, y_1, y_2 \in \mathbf{R}.$$

$$d_H(H(t, x_1, y_1), H(t, x_2, y_2)) \leq q(t)(|x_1 - x_2| + |y_1 - y_2|) \quad \forall\, x_1, x_2, y_1, y_2 \in \mathbf{R}.$$

Also, we use, next, the notations

$$K_1(t, s) = (1 - \sigma_1) + \sigma_1(t - s), \quad t, s \in I, \quad k_1 = (1 - \sigma_1) + \sigma_1 T,$$
$$K_2(t, s) = (1 - \sigma_2) + \sigma_2(t - s), \quad t, s \in I, \quad k_2 = (1 - \sigma_2) + \sigma_2 T,$$
$$l(t) = k_1 p(t) + k_2 q(t), \quad t \in I.$$

Theorem 3.1. *Assume that Hypothesis 3.1 is satisfied, $|l(.)|_1 < 1$, consider $(u(.), v(.)) \in AC(I, \mathbf{R}) \times AC(I, \mathbf{R})$ such that there exists $m(.), n(.) \in L^1(I, \mathbf{R})$ with $d(D_{CF}^{\sigma_1} u(t), G(t, u(t), v(t))) \leq m(t)$ a.e. $t \in I$, $u(0) = u_0$, $u'(0) = u_1$ and $d(D_{CF}^{\sigma_2} v(t), H(t, u(t), v(t))) \leq n(t)$ a.e. $t \in I$, $v(0) = v_0$, $v'(0) = v_1$.*

Then there exists $(y(.), z(.)) \in AC(I, \mathbf{R}) \times AC(I, \mathbf{R})$ a solution of problem (1.1) satisfying for all $t \in I$

$$|y(t) - u(t)| + |z(t) - v(t)| \leq (|y_0 - u_0| + T|y_1 - u_1| + |z_0 - v_0| + T|z_1 - v_1| + k_1|m(.)|_1 + k_2|n(.)|_1)(1 - |l(.)|_1)^{-1}. \tag{3.1}$$

Proof. We apply Lemma 2.2 with $F(t) = G(t, u(t), v(t))$, $a(t) = D_{CF}^{\sigma_1} u(t)$ and $b(t) = m(t)$, $t \in I$ to deduce the existence of a mesurable selection $g_1(t) \in G(t, u(t), v(t))$ *a.e.* $t \in I$ such that

$$|g_1(t) - D_{CF}^{\sigma_1} u(t)| \leq m(t) \quad a.e.\ (I).$$

Similarly, there exists $h_1(t) \in H(t, u(t), v(t))$ *a.e.* $t \in I$ such that $|h_1(t) - D_{CF}^{\sigma_2} v(t)| \leq n(t)$ *a.e.* (I).

Put $y_1(t) = y_0 + y_1 t + \int_0^t K_1(t, s) g_1(s) ds$, $z_1(t) = z_0 + z_1 t + \int_0^t K_2(t, s) h_1(s) ds$. One has

$$|y_1(t) - u(t)| = |y_0 + y_1 t - u_0 - u_1 t + \int_0^t K_1(t, s)(g_1(s) - D_{CF}^{\sigma_1} u(s)) ds|$$
$$\leq |y_0 - u_0| + T|y_1 - u_1| + k_1|m(.)|_1.$$

Similarly, $|z_1(t) - v(t)| \leq |z_0 - v_0| + T|z_1 - v_1| + k_2|n(.)|_1$ and therefore,

$$|y_1(t) - u(t)| + |z_1(t) - v(t)| \leq |y_0 - u_0| + T|y_1 - u_1| + k_1|m(.)|_1 + |z_0 - v_0| + T|z_1 - v_1| + k_2|n(.)|_1 =: K$$

Next, we will define, by induction the sequences $y_n(.), z_n(.) \in AC(I, \mathbf{R})$ and $g_n(.)$, $h_n(.) \in L^1(I, \mathbf{R})$, $n \geq 1$ such that

$$\begin{aligned} y_n(t) &= y_0 + y_1 t + \int_0^t K_1(t,s) g_n(s) ds \\ z_n(t) &= z_0 + z_1 t + \int_0^t K_2(t,s) h_n(s) ds \end{aligned} \tag{3.2}$$

$$g_n(t) \in G(t, u_{n-1}(t), v_{n-1}(t)),\ h_n(t) \in H(t, u_{n-1}(t), v_{n-1}(t)) \quad a.e.\,(I), \tag{3.3}$$

$$\begin{aligned} |g_{n+1}(t) - g_n(t)| &\leq p(t)(|y_n(t) - y_{n-1}(t)| + |z_n(t) - z_{n-1}(t)|) \quad a.e.\ (I), \\ |h_{n+1}(t) - h_n(t)| &\leq q(t)(|y_n(t) - y_{n-1}(t)| + |z_n(t) - z_{n-1}(t)|) \quad a.e.\ (I). \end{aligned} \tag{3.4}$$

We show, first, that from (3.2)-(3.4) it follows that

$$|y_{n+1}(t) - y_n(t)| + |z_{n+1}(t) - z_n(t)| \leq K(|l(.)|_1)^n \quad a.e.\ (I) \quad \forall n \in \mathbf{N}. \tag{3.5}$$

For $n = 0$ the inequality above was already proved. Assume that the last inequality is true for $n-1$. For almost all $t \in I$,

$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &\leq \int_0^T |K_1(t,s)|.|g_{n+1}(s) - g_n(s)| ds \leq k_1 \int_0^T |g_{n+1}(s) - g_n(s)| ds \\ &\leq k_1 \int_0^T p(s)(|y_n(s) - y_{n-1}(s)| + |z_n(s) - z_{n-1}(s)|) ds \leq k_1 K(|l|_1)^{n-1} \int_0^T p(s) ds. \end{aligned}$$

Similarly, for almost all $t \in I$,

$$|z_{n+1}(t) - z_n(t)| \leq k_2 K(|l(.)|_1)^{n-1} \int_0^T q(s) ds$$

and, therefore, for almost all $t \in I$ and for all $n \in \mathbf{N}$

$$|y_{n+1}(t) - y_n(t)| + |z_{n+1}(t) - z_n(t)| \leq K(k_1|p(.)|_1 + k_2|q(.)|_1)(|l(.)|_1)^{n-1} = K(|l(.)|_1)^n.$$

From (3.5) it follows that the sequences $\{y_n(.)\}, \{z_n(.)\}$ are Cauchy in the space $C(I, \mathbf{R})$. Consider $y(.) \in C(I, \mathbf{R})$ and $z(.) \in C(I, \mathbf{R})$ their uniform limits in $C(I, \mathbf{R})$. In particular, (3.4) gives that, for almost all $t \in I$, the sequences $\{g_n(t)\}, \{h_n(t)\}$ are Cauchy in $\mathbf{R}$. Set $g(.), h(.)$ their pointwise limit.

Taking into account Hypothesis 3.1 and relations (3.5) we obtain the estimates

$$\begin{aligned} &|y_n(t) - u(t)| + |z_n(t) - v(t)| \leq |y_1(t) - u(t)| + |z_1(t) - v(t)| + \\ &\sum_{i=1}^{n-1} (|y_{i+1}(t) - y_i(t)| + |z_{i+1}(t) - z_i(t)|) \leq K + \sum_{i=1}^{n} K(|l(.)|_1)^i \leq \tfrac{K}{1-|l(.)|_1}. \end{aligned} \tag{3.6}$$

$$\begin{aligned} &|g_n(t) - D_{CF}^{\sigma_1} u(t)| + |h_n(t) - D_{CF}^{\sigma_2} v(t)| \leq |g_1(t) - D_{CF}^{\sigma_1} u(t)| + |h_1(t) - \\ &D_{CF}^{\sigma_2} v(t)| + \sum_{i=1}^{n-1} (|g_{i+1}(t) - g_i(t)| + |h_{i+1}(t) - h_i(t)|) \leq |g_1(t) - D_{CF}^{\sigma_1} u(t)| \\ &+ |h_1(t) - D_{CF}^{\sigma_2} v(t)| + \sum_{i=1}^{n-1} (p(t) + q(t))(|y_i(t) - y_{i-1}(t)| + |z_i(t) - z_{i-1}(t)|) \\ &\leq m(t) + n(t) + (p(t) + q(t)) \tfrac{K}{1-|l(.)|_1} \end{aligned}$$

for almost all $t \in I$.

The last inequality shows that the sequences $g_n(.)$, $h_n(.)$ are integrably bounded and therefore, their limits $g(.), h(.)$ belong to $L^1(I, \mathbf{R})$.

If we let $n \to \infty$ in (3.2) we obtain that $(y(.), z(.))$ is a solution of problem (1.1). Also, if $n \to \infty$ in (3.6) we get (3.1).

The proof is finished if the construction in (3.2)-(3.4) is provided. This will be done, again, by induction. Assume that for $J \geq 1$, $y_j(.), z_j(.) \in C(I, \mathbf{R})$ and $g_j(.), h_j(.) \in L^1(I, \mathbf{R})$, $j = 1, 2, ...J$ with (3.2) and (3.4) for $n = 1, 2, ...J$ and (3.3) for $j = 1, 2, ...J-1$ are defined.

The set-valued maps $t \to G(t, y_J(t), z_J(t))$, $t \to H(t, y_J(t), z_J(t))$ are measurable; also the maps $t \to p(t)|y_J(t) - y_{J-1}(t)| + p(t)|z_J(t) - z_{J-1}(t)|$, $t \to q(t)|y_J(t) - y_{J-1}(t)| + q(t)|z_J(t) - z_{J-1}(t)|$ are measurable. By the lipschitzianity of $G(t,.,.)$ and

$H(t,.,.)$ we find

$$G(t,y_J(t),z_J(t))\cap\{g_J(t)+(p(t)|y_J(t)-y_{J-1}(t)|+p(t)|z_J(t)-z_{J-1}(t)|)[-1,1]\}\neq\emptyset,$$
$$H(t,y_J(t),z_J(t))\cap\{h_J(t)+(q(t)|y_J(t)-y_{J-1}(t)|+q(t)|z_J(t)-z_{J-1}(t)|)[-1,1]\}\neq\emptyset$$

for almost all $t\in I$.

So, we are able to apply Lemma 2.2 with $F(t)=G(t,y_J(t),z_J(t))$ (resp., $F(t)=H(t,y_J(t),z_J(t))$) $a(t)=g_J(t)$ (resp., $a(t)=h_J(t)$) and $b(t)=p(t)(|y_J(t)-y_{J-1}(t)|+|z_J(t)-z_{J-1}(t)|)$ (resp., $b(t)=q(t)(|y_J(t)-y_{J-1}(t)|+|z_J(t)-z_{J-1}(t)|)$) in order to deduce the existence of measurable selections $g_{J+1}(.)$ of $G(.,y_J(.),z_J(.))$ and $h_{J+1}(.)$ of $H(.,u_J(.),v_J(.))$ that satisfy

$$|g_{J+1}(t)-g_J(t)|\leq p(t)(|y_J(t)-y_{J-1}(t)|+|z_J(t)-z_{J-1}(t)|)\quad a.e.\ (I),$$
$$|h_{J+1}(t)-h_J(t)|\leq q(t)(|y_J(t)-y_{J-1}(t)|+|z_J(t)-z_{J-1}(t)|)\quad a.e.\ (I).$$

It remains to put $(y_{J+1}(.),z_{J+1}(.))$ as in (3.2) with $n=J+1$. □

Corollary 3.2. *Assume that Hypothesis 3.1 is satisfied,* $d(0,G(t,0,0))\leq p(t)$*,* $d(0,H(t,0,0))\leq q(t)$ *a.e.* $t\in I$ *and* $|l(.)|_1<1$.

Then there exists $(y(.),z(.))\in AC(I,\mathbf{R})\times AC(I,\mathbf{R})$ *a solution of problem (1.1) such that, for all* $t\in I$*,*

$$|y(t)|+|z(t)|\leq\frac{|y_0|+T|y_1|+|z_0|+T|z_1|+k_1|p(.)|_1+k_2|q(.)|_1}{(1-k_1|p(.)|_1-k_2|q(.)|_1}.$$

Proof. It is enough to apply Theorem 3.1 with $u(.)=v(.)=0$, $m(.)=p(.)$ and $n(.)=q(.)$. □

4. Conclusions and Discussions

In the present paper we extended existence results of Filippov type obtained for a fractional differential inclusion of Caputo-Fabrizio type to the more general problem of coupled system of such fractional differential inclusions. At the same time, the present paper may be regarded as a continuation of the study in [2] to the more general framework of fractional differential inclusions.

Existence results as in Corollary 3.2 above may be obtained, also, via a fixed point approach, namely; Covitz and Nadler set-valued contraction principle. We avoided these fixed point techniques because this approach requires that the values of $G(.,.)$ and $H(.,.)$ are compact and does not provides a priori bounds as in Corollary 3.2.

As potential directions for future works we note that Theorem 3.1 is an essential tool in order to obtain qualitative results concerning the solutions of the problems considered: controllability along a given solution and differentiability of trajectories with respect to the initial conditions.

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