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GENERALIZATIONS OF IMPLICATION-BASED FUZZY SUBALGEBRAS IN BCK/BCI-ALGEBRAS

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ABSTRACT. In [2], Jun discussed implication-based subalgebras in BCK/BCI-algebras. In this article, more general forms than Jun's results are discussed. We provide an example to show that a fuzzy subalgebra with thresholds 0 and 0.5 is not an implication-based subalgebra under the Łuckasiewicz implication operator, and then conditions for a fuzzy subalgebra with thresholds 0 and 0.5 to be an implication-based subalgebra under the Łuckasiewicz implication operator are provided.

1. INTRODUCTION

As a general form of fuzzy subalgebras in BCK/BCI-algebras, Jun [1, 2, 3] discussed fuzzy subalgebras with thresholds ε and δ in BCK/BCI-algebras, and also dealt with implication-based subalgebras in BCK/BCI-algebras. In this paper, we discuss more general forms than Jun's results. We provide an example to show that a fuzzy subalgebra with thresholds 0 and 0.5 is not an implication-based subalgebra under the Łuckasiewicz implication operator, and then we consider conditions for a fuzzy subalgebra with thresholds 0 and 0.5 to be an implication-based subalgebra under the Łuckasiewicz implication operator.

2. PRELIMINARIES

By a BCI-*algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(i) $(\forall x, y, z \in X)\,(((x * y) * (x * z)) * (z * y) = 0)$,
(ii) $(\forall x, y \in X)\,((x * (x * y)) * y = 0)$,
(iii) $(\forall x \in X)\,(x * x = 0)$,
(iv) $(\forall x, y \in X)\,(x * y = y * x = 0 \;\Rightarrow\; x = y)$.

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We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. In what follows let X denote a BCK/BCI-algebra unless otherwise specified. A nonempty subset S of X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. We refer the reader to the book [4] for further information regarding BCK/BCI-algebras.

A fuzzy set μ in a set X of the form

$$\mu(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [5] gave meaning to the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, \mathrm{q}, \in\vee\, q\,, \in\wedge\, q\,\}$.

To say that $x_t \in \mu$ (resp. $x_t \mathrm{q} \mu$) means that $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ.

To say that $x_t \in\vee\, q\ \mu$ (resp. $x_t \in\wedge\, q\ \mu$) means that $x_t \in \mu$ or $x_t \mathrm{q} \mu$ (resp. $x_t \in \mu$ and $x_t \mathrm{q} \mu$).

In what follows, let X denote a BCK/BCI-algebra unless otherwise specified.

A fuzzy set μ in X is called a *fuzzy subalgebra* of X if it satisfies

$$(\forall x, y \in X)\, (\mu(x * y) \geq m(\mu(x), \mu(y))). \tag{2.1}$$

Definition 2.1 ([1]). A fuzzy set μ in X is said to be an $(\in, \in\vee\, q\,)$*-fuzzy subalgebra* of X if it satisfies the following condition:

$$(\forall x, y \in X)(\forall t_1, t_2 \in (0,1]) \left(x_{t_1} \in \mu,\ y_{t_2} \in \mu\ \Rightarrow\ (x * y)_{\min\{t_1, t_2\}} \in\vee\, q\, \mu \right). \tag{2.2}$$

Definition 2.2 ([2]). A fuzzy set μ in X is called a *fuzzy subalgebra with thresholds ε and δ* of X, where $\varepsilon, \delta \in [0,1]$ with $\varepsilon < \delta$, if it satisfies the following condition:

$$(\forall x, y \in X)\, (\max\{\mu(x * y), \varepsilon\} \geq \min\{\mu(x), \mu(y), \delta\})\,. \tag{2.3}$$

3. Implication-based fuzzy subalgebras

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\wedge$, $\vee$, $\neg$, $\rightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe U of discourse, we display the fuzzy logical and corresponding set-theoretical notations used in this paper

$$[x \in \mu] = \mu(x), \tag{3.1}$$

$$[\Phi \wedge \Psi] = \min\{[\Phi], [\Psi]\}, \tag{3.2}$$

$$[\Phi \rightarrow \Psi] = \min\{1, 1 - [\Phi] + [\Psi]\}, \tag{3.3}$$

$$[\forall x \Phi(x)] = \inf_{x \in U} [\Phi(x)], \tag{3.4}$$

$$\models \Phi \text{ if and only if } [\Phi] = 1 \text{ for all valuations.} \tag{3.5}$$

The truth valuation rules given in (3.3) are those in the Łukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following.

(a) Gaines-Rescher implication operator (I_{GR}):

$$I_{\mathrm{GR}}(a, b) = \begin{cases} 1 & \text{if } a \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Gödel implication operator (I_{G}):

$$I_{\mathrm{G}}(a,b)=\begin{cases}1 & \text{if } a\leq b,\\ b & \text{otherwise.}\end{cases}$$

(c) The contraposition of Gödel implication operator (I_{cG}):

$$I_{\mathrm{cG}}(a,b)=\begin{cases}1 & \text{if } a\leq b,\\ 1-a & \text{otherwise.}\end{cases}$$

(d) The Łuckasiewicz implication operator (I_{LI}):

$$I_{\mathrm{LI}}(a,b)=\begin{cases}1 & \text{if } a\leq b,\\ 1-a+b & \text{otherwise.}\end{cases}$$

Ying [7] introduced the concept of fuzzifying topology. We can expand his/her idea to BCK/BCI-algebras, and we define a fuzzifying subalgebra as follows.

Definition 3.1. A fuzzy set μ in X is called a *fuzzifying subalgebra* of X if it satisfies the following condition:

$$\text{for any } x,y\in X, \models\ [x\in\mu]\wedge[y\in\mu]\rightarrow[x*y\in\mu]. \tag{3.6}$$

Obviously, the condition (3.6) is equivalent to the condition (2.1). Therefore a fuzzifying subalgebra is an ordinary fuzzy subalgebra. In [6], the concept of t-tautology is introduced, i.e.,

$$\models_t \Phi \text{ if and only if } [\Phi]\geq t \text{ for all valuations.} \tag{3.7}$$

Definition 3.2 ([2])**.** Let μ be a fuzzy set in X and $t\in(0,1]$. μ is called a *t-implication-based subalgebra* of X if it satisfies:

$$\text{for any } x,y\in X, \models_t\ [x\in\mu]\wedge[y\in\mu]\rightarrow[x*y\in\mu].$$

Let I be an implication operator. Clearly, μ is a t-implication-based subalgebra of X if and only if it satisfies

$$(\forall x,y\in X)\ (I(m(\mu(x),\mu(y)),\mu(x*y))\geq t).$$

Example 3.3 ([1])**.** Consider a BCI-algebra $X=\{0,a,b,c\}$ with the following Cayley table :

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let μ be a fuzzy set in X defined by $\mu(0)=0.6$, $\mu(a)=0.7$, and $\mu(b)=\mu(c)=0.3$. Then μ is a t-implication-based subalgebra of X for all $t\in(0,0.5]$ under the Gödel implication operator I_{G}. Also μ is a 0.4-implication-based subalgebra of X under the contraposition of Gödel implication operator I_{cG}.

Example 3.4. Consider a BCK-algebra $X=\{0,a,b,c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	a
c	c	c	c	0

Let μ be a fuzzy set in X defined by $\mu(0) = 0.5$, $\mu(a) = 0.4$, $\mu(b) = 0.7$ and $\mu(c) = 0.6$. By routine calculations, we know that μ is a t-implication-based subalgebra of X for all $t \in (0, 0.4]$ under the Gödel implication operator I_{G}.

Example 3.5. Consider the BCI-algebra $X = \{0, a, b, c\}$ in Example 3.3. Define a fuzzy set μ in X by $\mu(0) = 0.6$, $\mu(a) = 0.7$, $\mu(b) = 0.4$ and $\mu(c) = 0.2$. Then μ is not a fuzzy subalgebra of X since

$$\mu(a * b) = \mu(c) = 0.2 < 0.4 = \min\{\mu(a), \mu(b)\}.$$

By routine calculations, we know that μ is a t-implication-based fuzzy subalgebra of X for all $t \in (0, 0.8]$ under the Łuckasiewicz implication operator I_{LI}.

Note that if $t_1, t_2 \in (0, 1]$ with $t_1 > t_2$, then every t_1-implication-based subalgebra of X is a t_2-implication-based subalgebra of X. But the converse is false. In fact, in Example 3.4, the t-implication-based subalgebra of X for all $t \in (0, 0.4]$ under the Gödel implication operator I_{G} is not a t-implication-based subalgebra of X for $t \in (0.4, 1]$ under the Gödel implication operator I_{G} since

$$I_{\mathrm{G}}\left(\min\{\mu(b), \mu(c)\}, \mu(b * c)\right) = I_{\mathrm{G}}\left(0.6, 0.4\right) = 0.4 \ngeq t$$

for $t \in (0.4, 1]$.

Lemma 3.1 ([1]). *A fuzzy set μ in X is an $(\in, \in\vee q)$-fuzzy subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X)\left(\mu(x * y) \geq \min\{\mu(x), \mu(y), 0.5\}\right). \tag{3.8}$$

Theorem 3.2. *For any fuzzy set μ in X, if $I = I_{\mathrm{G}}$ and μ is an $(\in, \in\vee q)$-fuzzy subalgebra of X, then μ is a t-implication-based subalgebra of X for all $t \in (0, 0.5]$.*

Proof. Let $t \in (0, 0.5]$ and assume that μ is an $(\in, \in\vee q)$-fuzzy subalgebra of X. Then

$$\mu(x * y) \geq \min\{\mu(x), \mu(y), 0.5\}$$

for all $x, y \in X$. If $\min\{\mu(x), \mu(y)\} \leq 0.5$, then $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ and so

$$I_{\mathrm{G}}\left(\min\{\mu(x), \mu(y)\}, \mu(x * y)\right) = 1 \geq t$$

Now, suppose that $\min\{\mu(x), \mu(y)\} > 0.5$. Then $\mu(x * y) \geq 0.5$, and either $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ or $\mu(x * y) < \min\{\mu(x), \mu(y)\}$. If $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, then

$$I_{\mathrm{G}}\left(\min\{\mu(x), \mu(y)\}, \mu(x * y)\right) = 1 \geq t.$$

If $\mu(x * y) < \min\{\mu(x), \mu(y)\}$, then

$$I_{\mathrm{G}}\left(\min\{\mu(x), \mu(y)\}, \mu(x * y)\right) = \mu(x * y) \geq 0.5 \geq t.$$

Therefore μ is a t-implication-based subalgebra of X for all $t \in (0, 0.5]$. □

In Example 3.4, we have shown that the fuzzy set μ is a t-implication-based subalgebra of X for all $t \in (0, 0.4]$ under the Gödel implication operator I_{G}. But μ is not an $(\in, \in\vee q)$-fuzzy subalgebra of X. This shows that the partial converse of Theorem 3.2 is not true.

Corollary 3.3. *For any fuzzy set μ in X, if the set*

$$A := \{x \in X \mid \mu(x) \geq t\}$$

is a subalgebra of X for all $t \in (0, 0.5]$, then μ is a t-implication-based subalgebra of X for all $t \in (0, 0.5]$ under the Gödel implication operator.

Proof. The proof is straightforward. □

Corollary 3.4. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras.*

(1) *If ν is an $(\in, \in\vee q)$-fuzzy subalgebra of Y, then its inverse image $f^{-1}(\nu)$ under f is a t-implication-based subalgebra of X for all $t \in (0, 0.5]$ under the the Gödel implication operator I_{G}.*

(2) *If f is onto and μ is an $(\in, \in\vee q)$-fuzzy subalgebra of X with the sup property, then its image $f(\mu)$ under f is a t-implication-based subalgebra of X for all $t \in (0, 0.5]$ under the the Gödel implication operator I_{G}.*

Proof. Note that $f^{-1}(\nu)$ and $f(\mu)$ are $(\in, \in\vee q)$-fuzzy subalgebras of X and Y respectively (see [3, Theorem 3.6]). By Theorem 3.2, we know that $f^{-1}(\nu)$ and $f(\mu)$ are t-implication-based subalgebras of X and Y, respectively, for all $t \in (0, 0.5]$ under the Gödel implication operator I_{G}. □

Theorem 3.5. *Consider $I = I_{\mathrm{G}}$ and let $t \in [0.5, 1]$. If μ is a t-implication-based subalgebra of X, then μ is an $(\in, \in\vee q)$-fuzzy subalgebra of X.*

Proof. Let $t \in [0.5, 1]$ be such that μ is a t-implication-based subalgebra of X. Then

$$I_{\mathrm{G}}(\min\{\mu(x), \mu(y)\}, \mu(x * y)) \geq t$$

for all $x, y \in X$, and so either $I_{\mathrm{G}}(\min\{\mu(x), \mu(y)\}, \mu(x * y)) = 1$, that is,

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$$

or $I_{\mathrm{G}}(\min\{\mu(x), \mu(y)\}, \mu(x * y)) = \mu(x * y) \geq t \geq 0.5$. Hence

$$\mu(x * y) \geq \min\{\mu(x), \mu(y), 0.5\}.$$

Using Lemma 3.1, we know that μ is an $(\in, \in\vee q)$-fuzzy subalgebra of X. □

Corollary 3.6. *For any $t \in [0.5, 1]$, if μ is a t-implication-based subalgebra of X under Gödel implication operator I_{G}, then μ is fuzzy subalgebra of X with thresholds $\varepsilon = 0$ and $\delta \in (0, 0.5]$.*

Proof. The proof is straightforward. □

Corollary 3.7. *For any $t \in [0.5, 1]$, if μ is a t-implication-based subalgebra of X under Gödel implication operator I_{G}, then the set*

$$A_k := \{x \in X \mid \mu(x) \geq k\}$$

is a subalgebra of X for all $k \in (0, 0.5]$.

Proof. The proof is straightforward. □

Combining Theorems 3.2 and 3.5, we have the following corollary.

Corollary 3.8 ([2]). *For any fuzzy set μ in X, if $I = I_{\mathrm{G}}$, then μ is a 0.5-implication-based fuzzy subalgebra of X if and only if μ is a fuzzy subalgebra of X with thresholds $\varepsilon = 0$ and $\delta = 0.5$, that is, μ is an $(\in, \in\vee q)$-fuzzy subalgebra of X.*

The following example shows that for every fuzzy set μ in X, there exists a subinterval (α, β) of the interval $[0.5, 1]$ such that μ is an $(\in, \in\vee q)$-fuzzy subalgebra of X which is not a t-implication-based subalgebra of X for any $t \in (\alpha, \beta)$ under the Gödel implication operator I_{G}.

TABLE 1. Calculations of $R := I_{\mathrm{G}}\left(\min\{\mu(x), \mu(y)\}, \mu(x * y)\right)$.

x	y	$\mu(x)$	$\mu(y)$	$\min\{\mu(x), \mu(y)\}$	$x * y$	$\mu(x * y)$	R
0	0	0.9	0.9	0.9	0	0.9	1
0	1	0.9	0.8	0.8	0	0.9	1
0	a	0.9	0.5	0.5	c	0.6	1
0	b	0.9	0.7	0.7	c	0.6	0.6
0	c	0.9	0.6	0.6	a	0.5	0.5
1	0	0.8	0.9	0.8	1	0.8	1
1	1	0.8	0.8	0.8	0	0.9	1
1	a	0.8	0.5	0.5	c	0.6	1
1	b	0.8	0.7	0.7	c	0.6	0.6
1	c	0.8	0.6	0.6	a	0.5	0.5
a	0	0.5	0.9	0.5	a	0.5	1
a	1	0.5	0.8	0.5	a	0.5	1
a	a	0.5	0.5	0.5	0	0.9	1
a	b	0.5	0.7	0.5	0	0.9	1
a	c	0.5	0.6	0.5	c	0.6	1
b	0	0.7	0.9	0.7	b	0.7	1
b	1	0.7	0.8	0.7	a	0.5	0.5
b	a	0.7	0.5	0.5	1	0.8	1
b	b	0.7	0.7	0.7	0	0.9	1
b	c	0.7	0.6	0.6	c	0.6	1
c	0	0.6	0.9	0.6	c	0.6	1
c	1	0.6	0.8	0.6	c	0.6	1
c	a	0.6	0.5	0.5	a	0.5	1
c	b	0.6	0.7	0.6	a	0.5	0.5
c	c	0.6	0.6	0.6	0	0.9	1

Example 3.6. Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the following Cayley table:

$$
\begin{array}{c|ccccc}
* & 0 & 1 & a & b & c \\
\hline
0 & 0 & 0 & c & c & a \\
1 & 1 & 0 & c & c & a \\
a & a & a & 0 & 0 & c \\
b & b & a & 1 & 0 & c \\
c & c & c & a & a & 0
\end{array}
$$

Define a fuzzy set μ in X by

$$
\mu : X \to [0,1],\ x \mapsto \begin{cases} 0.9 & \text{if } x = 0, \\ 0.8 & \text{if } x = 1, \\ 0.5 & \text{if } x = a, \\ 0.7 & \text{if } x = b, \\ 0.6 & \text{if } x = c. \end{cases}
$$

Then μ is an $(\in, \in\vee\, q)$-fuzzy subalgebra of X which is not a fuzzy subalgebra of X, and the calculation of $I_{\mathrm{G}}\left(\min\{\mu(x), \mu(y)\}, \mu(x * y)\right)$ is given by Table 1.

We know that μ is not a t-implication-based subalgebra of X for any $t \in (0.5, 1]$ under the Gödel implication operator I_{G} since

$$I_{\mathrm{G}}\left(\min\{\mu(1), \mu(c)\}, \mu(1 * c)\right) = I_{\mathrm{G}}\left(0.6, 0.5\right) = 0.5 \ngeq t$$

for any $t \in (0.5, 1]$.

Theorem 3.9. *Consider $I = I_{\mathrm{cG}}$ and let $t \in [0.5, 1]$. If μ is a t-implication-based subalgebra of X, then μ is a fuzzy subalgebra of X with thresholds $\varepsilon = t$ and δ, where $\delta = \sup_{x \in X} \mu(x)$.*

Proof. Let $t \in [0.5, 1]$ and assume that μ is a t-implication-based subalgebra of X. Then

$$I_{\mathrm{cG}}\left(\min\{\mu(x), \mu(y)\}, \mu(x * y)\right) \geq t$$

for all $x, y \in X$, and so either $I_{\mathrm{cG}}\left(\min\{\mu(x), \mu(y)\}, \mu(x * y)\right) = 1$, that is,

$$\min\{\mu(x), \mu(y)\} \leq \mu(x * y)$$

or $1 - \min\{\mu(x), \mu(y)\} = I_{\mathrm{cG}}\left(\min\{\mu(x), \mu(y)\}, \mu(x * y)\right) \geq t$, that is,

$$\min\{\mu(x), \mu(y)\} \leq 1 - t \leq t$$

since $t \in [0.5, 1]$. It follows that

$$\max\{\mu(x * y), t\} \geq \min\{\mu(x), \mu(y)\} = \min\{\mu(x), \mu(y), \delta\}.$$

Therefore μ is a fuzzy subalgebra of X with thresholds $\varepsilon = t$ and $\delta = \sup_{x \in X} \mu(x)$. □

Theorem 3.10. *Consider $I = I_{\mathrm{cG}}$ and let μ be a fuzzy set in X. For every $t \in (0, 0.5]$, if μ is a t-implication-based subalgebra of X, then μ is a fuzzy subalgebra of X with thresholds $\varepsilon = 1 - t$ and $\delta = \sup_{x \in X} \mu(x)$.*

Proof. Assume that μ is a t-implication-based subalgebra of X for $t \in (0, 0.5]$. Then

$$I_{\mathrm{cG}}\left(\min\{\mu(x), \mu(y)\}, \mu(x * y)\right) \geq t,$$

which implies that either $\min\{\mu(x), \mu(y)\} \leq \mu(x * y)$ or

$$1 - \min\{\mu(x), \mu(y)\} = I_{\mathrm{cG}}\left(\min\{\mu(x), \mu(y)\}, \mu(x * y)\right) \geq t,$$

and so $\min\{\mu(x), \mu(y)\} \leq 1 - t$. It follows that

$$\max\{\mu(x * y), 1 - t\} \geq \min\{\mu(x), \mu(y)\} = \min\{\mu(x), \mu(y), \delta\}.$$

Therefore μ is a fuzzy subalgebra of X with thresholds $\varepsilon = 1 - t$ and $\delta = \sup_{x \in X} \mu(x)$. □

Corollary 3.11. *For every $t \in (0, 0.5]$, if μ is a t-implication-based subalgebra of X under the contraposition of Gödel implication operator I_{cG}, then μ is a fuzzy subalgebra of X with thresholds $\varepsilon = 1 - t$ and $\delta = 1$.*

For the converse of Theorem 3.9, we have the following theorem.

Theorem 3.12. *Consider $I = I_{\mathrm{cG}}$ and let μ be a fuzzy set in X. For every $t \in (0, 0.5]$, if μ is a fuzzy subalgebra of X with thresholds $\varepsilon = t$ and $\delta = \sup_{x \in X} \mu(x)$, then μ is a t-implication-based subalgebra of X.*

Proof. Let $t \in (0, 0.5]$ and suppose that μ is a fuzzy subalgebra of X with thresholds $\varepsilon = t$ and $\delta = \sup_{x \in X} \mu(x)$. Then

$$\max\{\mu(x * y), t\} \geq \min\{\mu(x), \mu(y), \delta\} = \min\{\mu(x), \mu(y)\}.$$

If $\mu(x*y) \geq t$, then $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$ and so

$$I_{\mathrm{cG}}\left(\min\{\mu(x), \mu(y)\}, \mu(x*y)\right) = 1 \geq t.$$

If $\mu(x*y) < t$, then $\min\{\mu(x), \mu(y)\} \leq t$. Hence if $\min\{\mu(x), \mu(y)\} \leq \mu(x*y)$ then

$$I_{\mathrm{cG}}\left(\min\{\mu(x), \mu(y)\}, \mu(x*y)\right) = 1 \geq t.$$

If $\min\{\mu(x), \mu(y)\} > \mu(x*y)$, then

$$I_{\mathrm{cG}}\left(\min\{\mu(x), \mu(y)\}, \mu(x*y)\right) = 1 - \min\{\mu(x), \mu(y)\} \geq 1 - t \geq t.$$

Consequently, μ is a t-implication-based subalgebra of X for every $t \in (0, 0.5]$. □

Corollary 3.13. *For every $t \in (0, 0.5]$, if μ is a fuzzy subalgebra of X with thresholds $\varepsilon = t$ and $\delta = 1$, then μ is a t-implication-based subalgebra of X under the contraposition of Gödel implication operator I_{cG}.*

Combining Corollaries 3.11 and 3.13, we have the following corollary.

Corollary 3.14 ([2])**.** *For any fuzzy set μ in X, if $I = I_{\mathrm{cG}}$, then μ is a 0.5-implication-based fuzzy subalgebra of X if and only if μ is a fuzzy subalgebra of X with thresholds $\varepsilon = 0.5$ and $\delta = 1$.*

Theorem 3.15. *Consider $I = I_{\mathrm{GR}}$ and let $t \in (0, 1]$. If μ is a t-implication-based subalgebra of X, then μ is a fuzzy subalgebra of X.*

Proof. Let $t \in (0, 1]$ be such that μ is a t-implication-based subalgebra of X under the Gaines-Rescher implication operator I_{GR}. Then

$$I_{\mathrm{GR}}\left(\min\{\mu(x), \mu(y)\}, \mu(x*y)\right) \geq t.$$

Since $t \neq 0$, it follows that $I_{\mathrm{GR}}\left(\min\{\mu(x), \mu(y)\}, \mu(x*y)\right) = 1$ and so that $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$. Therefore μ is a subalgebra of X. □

Corollary 3.16. *For any $t \in (0, 1]$, if μ is a t-implication-based subalgebra of X under the Gaines-Rescher implication operator I_{GR}, then the set*

$$A := \{x \in X \mid \mu(x) \geq t\}$$

is a subalgebra of X for all $t \in (0, 1]$.

Proof. The proof is straightforward. □

Theorem 3.17. *Every fuzzy subalgebra of X is a t-implication-based subalgebra of X for all $t \in (0, 1]$ under the Gaines-Rescher implication operator I_{GR}.*

Proof. The proof is straightforward. □

The following corollary is by Theorems 3.15 and 3.17.

Corollary 3.18 ([2])**.** *A fuzzy set in X is a 0.5-implication-based subalgebra of X under the Gaines-Rescher implication operator I_{GR} if and only if it is a fuzzy subalgebra of X.*

Theorem 3.19. *Every fuzzy subalgebra of X is a t-implication-based subalgebra of X for all $t \in (0, 1]$ under the Łuckasiewicz implication operator I_{LI}.*

Proof. The proof is straightforward. □

The following example shows that for a fuzzy set μ in X there exists $t \in (0, 1]$ such that

(1) μ is an $(\in, \in\vee\, q)$-fuzzy subalgebra of X.

(2) μ is not a t-implication-based subalgebra of X under the Łuckasiewicz implication operator I_{LI}.

Example 3.7. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table :

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Let μ be a fuzzy set in X defined by $\mu(0) = 0.6$, $\mu(a) = 0.5$, and $\mu(b) = 0.8$ and $\mu(c) = 0.9$. Then μ is an $(\in, \in\vee q)$-fuzzy subalgebra of X which is not a fuzzy subalgebra of X. But μ is not a 0.8-implication-based subalgebra of X under the Łuckasiewicz implication operator I_{LI} since

$$I_{\mathrm{LI}}\left(\min\{\mu(c), \mu(b)\}, \mu(c * b)\right) = I_{\mathrm{LI}}\left(0.8, 0.5\right) = 1 - 0.8 + 0.5 = 0.7 \ngeq 0.8.$$

We provide conditions for an $(\in, \in\vee q)$-fuzzy subalgebra of X to be a t-implication-based subalgebra of X under the Łuckasiewicz implication operator I_{LI}.

Theorem 3.20. *Let μ be an $(\in, \in\vee q)$-fuzzy subalgebra of X such that*

$$\min\{\mu(x), \mu(y)\} > \mu(x * y)$$

for some $x, y \in X$, and let

$$\mathcal{B} = \{(x, y) \in X \times X \mid \min\{\mu(x), \mu(y)\} > \mu(x * y)\}.$$

For any $(x, y) \in \mathcal{B}$, let $\omega_{(x,y)} = 1 - \min\{\mu(x), \mu(y)\} + \mu(x * y)$ and $\omega = \inf\limits_{(x,y)\in\mathcal{B}} \omega_{(x,y)}$. Then μ is a t-implication-based subalgebra of X for all $t \in (0, \omega]$ under the Łuckasiewicz implication operator I_{LI}.

Proof. If μ is an $(\in, \in\vee q)$-fuzzy subalgebra of X, then $\mu(x * y) \geq \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in X$. Suppose that $\min\{\mu(x), \mu(y)\} \leq 0.5$. Then $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$, and so

$$I_{\mathrm{LI}}\left(\min\{\mu(x), \mu(y)\}, \mu(x * y)\right) = 1 \geq t$$

for all $t \in (0, \omega]$. Assume that $\min\{\mu(x), \mu(y)\} > 0.5$ for all $x, y \in X$. Then $\mu(x * y) \geq 0.5$. Thus we have two cases:

(i) $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$,

(ii) $\mu(x * y) < \min\{\mu(x), \mu(y)\}$.

First case implies that

$$I_{\mathrm{LI}}\left(\min\{\mu(x), \mu(y)\}, \mu(x * y)\right) = 1 \geq t$$

for all $t \in (0, \omega]$. The second case induces

$$\begin{aligned} &I_{\mathrm{LI}}\left(\min\{\mu(x), \mu(y)\}, \mu(x * y)\right) \\ &= 1 - \min\{\mu(x), \mu(y)\} + \mu(x * y) \\ &= \omega_{(x,y)} \geq t \end{aligned}$$

for all $t \in (0, \omega]$. Therefore μ is a t-implication-based subalgebra of X for all $t \in (0, \omega]$ under the Łuckasiewicz implication operator I_{LI}. □

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