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FUZZY STABILITY FOR FINITE VARIABLE ADDITIVE FUNCTIONAL EQUATION IN CLASSICAL METHODS

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ABSTRACT. We examine the Ulam-Hyers stability of finite variable additive functional equation in fuzzy normed space using classical methods.

1. INTRODUCTION

The functional equation

$$f(x+y)=f(x)+f(y) \tag{1.1}$$

is referred to as the Cauchy additive functional equation and it's miles the most well-known functional equation. due to the fact $f(x)=kx$ is the solution of the functional equation (1.1), each solution of the additive equation is known as an additive function. Later, several researchers fascinated to concentrate the functional equations in fuzzy normed spaces (Ref [2, 4, 5, 8, 11]).

A whole lot of influence of Ulam, Hyers and Rassias on the improvement of stability issues of functional equations, the stability phenomenon that become proved by way of Th. M. Rassias is referred to as Hyers-Ulam-Rassias stability. The look at of stability problems for functional equations is associated with a query of Ulam regarding the stability of group homomorphisms and affirmatively spoke back for Banach spaces by way of Hyers. A generalized model of the concept of Hyers for about linear mappings turned into given with the aid of Th.M. Rassias.

The stability issues of numerous functional equations have been appreciably investigated with the aid of a number of authors [1, 3, 6, 7, 9, 10, 13] and there are many interesting consequences regarding this problem. The stability problems of numerous quadratic functional equations have been notably investigated by using some of authors and there are numerous interesting consequences regarding this problem M. Arunkumar. Because then, the stability issues of various functional equation had been appreciably investigated with the aid of a number of authors.

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In this paper, we investigate the generalized Ulam-Hyers stability of finite variable additive functional equation

$$\phi\left(\sum_{a=1}^{p} rt_a\right) + \sum_{j=1}^{p} \phi\left(-rt_b + \sum_{a=1;a\neq b}^{p} rt_a\right) = (p-1)\big[\sum_{a=1}^{p}(2a-1)\phi(t_a)\big] \tag{1.2}$$

where n is the positive integer with $\mathbb{N} - 0, 1, 2$ and k is the only odd positive integers, in Fuzzy Normed Space using different methods.

Definition 1.1. Let E be a real vector space. A function $N_n : E \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on E if $\forall\, a, b \in E$ and all $p, q \in \mathbb{R}$,

(N_1) $N_n(a, q) = 0$ for $q \leq 0$;
(N_2) $a = 0 \Leftrightarrow N_n(a, q) = 1\ \forall q > 0$;
(N_3) $N_n(\alpha a, q) = N_n(a, \frac{q}{|\alpha|})$ if $\alpha \neq 0$;
(N_4) $N_n(a + b, p + q) \geq \min\{N_n(a, p), N_n(b, q)\}$;
(N_5) $N_n(a, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{q\to\infty} N_n(a, q) = 1$;
(N_6) for $a \neq 0$, $N_n(a, \cdot)$ is continuous on $\mathbb{R}$.

The pair (E, N_n) is called a *fuzzy normed vector space*.

Theorem 1.1 (The Alternative of fixed point)**.** *Suppose that for a complete generalized metric space (E, d) and a strictly contractive mapping $\Omega : E \to E$ with Lipschitz constant L. Then, for each $a \in E$ either*
(B1) $d(\Omega^i a, \Omega^{i+1} a) = +\infty, \forall i \geq 0$, or
(B2) There exists natural number i_0 such that

i) $d(\Omega^i a, \Omega^{i+1} a) < \infty\ \ \forall i \geq i_0$;
ii) The sequence $(\Omega^i a)$ is convergent to a fixed point α^* of Ω;
iii) α^* is the unique fixed point of Ω in the set $F = \{\alpha \in E; d(\Omega^{i_0} a, \alpha) < \infty\}$;
iv) $d(\alpha^*, \alpha) \leq \frac{1}{1-L} d(\alpha, \Omega\alpha)\ \forall \alpha \in F$.

All over the upcoming sections, consider E, (Z, V) and (F, W) are Linear space, Fuzzy Normed space and Fuzzy Banach space respectively. And let us define a mapping $\phi : E \to F$ by

$$D\phi(t_1, t_2, \cdots, t_p) = \phi\left(\sum_{a=1}^{p} rt_a\right) + \sum_{j=1}^{p} \phi\left(-rt_b + \sum_{a=1;a\neq b}^{p} rt_a\right) - (p-1)\big[\sum_{a=1}^{p}(2a-1)\phi(t_a)\big]$$

for all $t_1, t_2, \cdots, t_p \in E$.

2. STABILITY RESULTS FOR (1.2): DIRECT METHOD

Theorem 2.1. *Let $\varsigma \in \{-1, 1\}$ be fixed and let $\Theta : E^p \to Z$ be a mapping such that for some $\alpha > 0$ with $\left(\frac{\alpha}{3}\right)^{\varsigma} < 1$*

$$V\left(\Theta\left(3^{\varsigma}t, 3^{\varsigma}t, \cdots, 3^{\varsigma}t\right), \zeta\right) \geq V\left(\alpha^{\varsigma}\Theta\left(0, t, 0, \cdots, 0\right), \zeta\right) \quad \forall t \in E,\ \zeta > 0 \tag{2.1}$$

and

$$\lim_{p\to\infty} V\left(\Theta\left(3^{\varsigma p}t_1, 3^{\varsigma p}t_2, \cdots, 3^{\varsigma p}t_p\right), 3^{\varsigma p}\zeta\right) = 1 \quad \forall t_1, t_2, \cdots, t_p \in E, \zeta > 0.$$

Suppose an odd mapping $\phi : E \to F$ fulfils the inequality

$$W\left(D\phi\left(t_1, t_2, \cdots, t_p\right), \zeta\right) \geq V\left(\Theta\left(t_1, t_2, \cdots, t_p\right), \zeta\right) \quad \forall t_1, t_2, \cdots, t_p \in E, \zeta > 0. \tag{2.2}$$

Then the limit

$$A_1(t) = W - \lim_{p\to\infty} \frac{\phi\left(3^{\varsigma p} t\right)}{3^{\varsigma p}}$$

exists for all $t \in E$ and $A_1 : E \to F$ is a unique additive mapping such that

$$W\left(\phi(t) - A_1(t), \zeta\right) \geq V\left(\Theta\left(0, t, 0, \cdots, 0\right), (p-1)\zeta \mid 3-\alpha \mid\right) \quad \forall t \in E,\ \zeta > 0. \tag{2.3}$$

Proof. Consider the case for $\varsigma = 1$. Switching $(t_1, t_2, \cdots, t_p)$ by $(0, t, 0, \cdots, 0)$ in (2.2), we obtain

$$W\left((p-1)\phi(3t) - 3(p-1)\phi(t), \zeta\right) \geq V\left(\Theta(0, t, 0, \cdots, 0), \zeta\right) \quad \forall t \in E, \zeta > 0. \tag{2.4}$$

From that (2.4)

$$W\left(\frac{\phi(3t)}{3} - \phi(t), \frac{\zeta}{3(p-1)}\right) \geq V\left(\Theta(0, t, 0, \cdots, 0), \zeta\right) \quad \forall t \in E, \zeta > 0. \tag{2.5}$$

Replacing t by $3^p t$ in (2.5), we reach

$$W\left(\frac{\phi(3^{p+1}t)}{3} - \phi(3^p t), \frac{\zeta}{3(p-1)}\right) \geq V\left(\Theta(0, 3^p t, 0 \cdots, 0), \zeta\right) \quad \forall t \in E, \zeta > 0. \tag{2.6}$$

Utilizing (2.1), (N_3) in (2.6) we have

$$W\left(\frac{\phi(3^{p+1}t)}{3} - \phi(3^p t), \frac{\zeta}{3(p-1)}\right) \geq V\left(\Theta(0, t, 0, \cdots, 0), \frac{\zeta}{\alpha^p}\right) \quad \forall t \in E, \zeta > 0. \tag{2.7}$$

It is easy to prove from (2.7), that

$$W\left(\frac{\phi(3^{(p+1)}t)}{3^{(p+1)}} - \frac{\phi(3^p t)}{3^p}, \frac{\zeta}{(p-1)(3^{(p+1)})}\right) \geq V\left(\Theta(0, t, 0, \cdots, 0), \frac{\zeta}{\alpha^p}\right) \tag{2.8}$$

holds for all $t \in E$ and all $\zeta > 0$. Interchanging ζ by $\alpha^p \zeta$ in (2.8), we have

$$W\left(\frac{\phi(3^{p+1}t)}{3^{(p+1)}} - \frac{\phi(3^p t)}{3^p}, \frac{\alpha^p \zeta}{3^{(p+1)}(p-1)}\right) \geq V\left(\Theta(0, t, 0, \cdots, 0), \zeta\right) \quad \forall t \in E, \zeta > 0. \tag{2.9}$$

It is easy to see that

$$\frac{\phi(3^p t)}{3^p} - \phi(t) = \sum_{a=0}^{p-1} \frac{\phi(3^{a+1}t)}{3^{(a+1)}} - \frac{\phi(3^a t)}{3^a} \quad \forall t \in E. \tag{2.10}$$

From (2.9) and (2.10), we attain

$$\begin{aligned} &W\left(\frac{\phi(3^p t)}{3^p} - \phi(t), \sum_{a=0}^{p-1} \frac{\zeta \alpha^a}{(3^{(a+1)}(p-1))}\right) \\ &\geq \min\{W\left(\frac{\phi(3^{a+1}t)}{3^{(a+1)}} - \frac{\phi(3^a t)}{3^a}, \frac{\zeta \alpha^a}{3^{(a+1)}(p-1)}\right) \ : a = 0, 1, \cdots, p-1\} \\ &\geq V\left(\Theta\left(0, t, 0 \cdots, 0\right), \zeta\right) \quad \forall t \in E, \zeta > 0. \end{aligned} \tag{2.11}$$

Switching t by $3^q t$ in (2.11) and utilizing (2.1), (N_3), we obtain

$$W\left(\frac{\phi(3^{p+q}t)}{3^{(p+q)}} - \frac{\phi(3^q t)}{3^q}, \sum_{a=0}^{p-1} \frac{\zeta \alpha^a}{3^{(a+1)}(p-1)}\right) \geq V\left(\Theta(0, 3^q t, 0, \cdots, 0), \zeta\right)$$

$$\geq V\left(\Theta(0,t,0,\cdots,0),\tfrac{\zeta}{\alpha^q}\right)$$

and so

$$W\left(\frac{\phi(3^{p+q}t)}{3^{(p+q)}}-\frac{\phi(3^q t)}{3^q},\sum_{a=q}^{p+q-1}\frac{\zeta\alpha^a}{3^{(a+1)}(p-1)}\right)\geq V\left(\Theta(0,t,0,\cdots,0),\zeta\right) \tag{2.12}$$

for all $t\in E$, $\zeta>0$ and all $q,p\geq 0$. Interchanging ζ by $\frac{\zeta}{\sum_{a=q}^{p+q-1}\frac{\alpha^a}{3^{(a+1)}(p-1)}}$ in (2.12), we reach

$$W\left(\frac{\phi(3^{p+q}t)}{3^{(p+q)}}-\frac{\phi(3^q t)}{3^q},\zeta\right)\geq V\left(\Theta(0,t,0,\cdots,0),\frac{\zeta}{\sum_{a=q}^{p+q-1}\frac{\alpha^a}{3^{(a+1)}(p-1)}}\right)\quad\forall t\in E,\zeta>0. \tag{2.13}$$

and all $q,p\geq 0$. Since $0<\alpha<3$ and $\sum_{a=0}^{p}\left(\frac{\alpha}{3}\right)^a<\infty$, the Cauchy criterion for convergence and (N_5) implies that $\{\frac{\phi(3^p t)}{3^p}\}$ is a Cauchy sequence in (F,W). Since (F,W) is a fuzzy Banach space, this sequence converges to some point $A_1(t)\in F$. So one can define the mapping $A_1:E\to F$ by

$$A_1(t):=W-\lim_{p\to\infty}\frac{\phi(3^p t)}{3^p}\quad\forall t\in E.$$

Since ϕ is odd. Letting $q=0$ in (2.13), we obtain

$$W\left(\frac{\phi(3^p t)}{3^p}-\phi(t),\zeta\right)\geq V\left(\Theta(0,t,0,\cdots,0),\frac{\zeta}{\sum_{a=0}^{p-1}\frac{\alpha^a}{3^{(a+1)}(p-1)}}\right)\quad\forall t\in E,\zeta>0. \tag{2.14}$$

Taking the limit as $p\to\infty$ in (2.14) and utilizing (N_6), we have

$$W\left(\phi(t)-A_1(t),\zeta\right)\geq V\left(\Theta(0,t,0,\cdots,0),(p-1)\zeta(3-\alpha)\right)\quad\forall t\in E,\zeta>0.$$

Now, we show that A_1 is additive. Switching $(t_1,t_2,\cdots,t_p)$ through $(3^p t_1,3^p t_2,\cdots,3^p t_p)$ in (2.2) respectively, we obtain

$$W\left(\frac{1}{3^p}D\phi\left(3^p t_1,3^p t_2,\cdots,3^p t_p\right),\zeta\right)\geq V\left(\Theta(3^p t_1,3^p t_2,\cdots,3^p t_p),3^p\zeta\right)\quad\forall t\in E,\zeta>0.$$

Since,

$$\lim_{p\to\infty}V\left(\Theta(3^p t_1,3^p t_2,\cdots,3^p t_p),3^p\zeta\right)=1.$$

A_1 fulfils (1.2). Hence $A_1:E\to F$ is additive. Now, we have to show that the uniqueness of A_1, consider $A_2:E\to F$ be another additive mapping fulfilling (2.3). Fix $t\in E$, clearly $A_1(3^p t)=3^p A_1(t)$ and $A_2(3^p t)=3^p A_2(t)\quad\forall\, t\in E,\ p\in\mathbb{N}$. From (2.3), we have

$$\begin{aligned}W(A_1(t)-A_2(t),\zeta)&=W\left(\tfrac{A_1(3^p t)}{3^p}-\tfrac{A_2(3^p t)}{3^p},\zeta\right)\\&\geq V\left(\Theta(0,t,0\cdots,0),\tfrac{(3^p)(p-1)\zeta(3-\alpha)}{2\alpha^p}\right)\quad\forall t\in E,\zeta>0.\end{aligned}$$

Since $\lim_{p\to\infty}\frac{(3^p)(p-1)\zeta(3-\alpha)}{2\alpha^p}=\infty$, we have

$$\lim_{p\to\infty}V\left(\Theta(0,t,0,\cdots,0),\frac{(3^p)(p-1)\zeta(3-\alpha)}{2\alpha^p}\right)=1.$$

Thus, $W(A_1(t)-A_2(t),\zeta)=1\quad\forall t\in E,\ \zeta>0$, and so $A_1(t)=A_2(t)$. For $\varsigma=-1$, we can derive the stability results by similar manner. $\square$

Corollary 2.2. *Suppose that the function $\phi : E \to F$ fulfils the inequality*

$$W(D\phi(t_1,t_2,\cdots,t_p),\zeta) \geq \begin{cases} V(\lambda,\zeta), \\ V(\lambda\sum_{a=1}^{p}||t_a||^{\psi},\zeta), \\ V(\lambda(\sum_{a=1}^{p}||t_a||^{p\psi}+\prod_{a=1}^{p}||t_a||^{\psi}),\zeta), \end{cases}$$

for all $t_1,t_2,\cdots,t_p \in E$ and all $\zeta > 0$, where λ,ψ are constants with $\lambda > 0$. Then there exists a unique additive mapping $A_1 : E \to F$ such that

$$W(\phi(t)-A_1(t),\zeta) \geq \begin{cases} V(\lambda,(p-1) \mid 2 \mid \zeta), \\ V\left(\lambda||t||^{\psi},(p-1) \mid 3-3^{\psi} \mid \zeta\right); & \psi \neq 1, \\ V\left(\lambda||t||^{p\psi},(p-1) \mid 3-3^{p\psi} \mid \zeta\right); & \psi \neq \frac{1}{p}, \end{cases}$$

for all $t \in E$ and all $\zeta > 0$.

3. Stability results for (1.2):fixed point method

For to prove the stability result, we define η_a is a constant such that

$$\eta_a = \begin{cases} 3 & if \quad a=0; \\ \frac{1}{3} & if \quad a=1; \end{cases}$$

and χ is the set such that $\chi = \{s/s : E \to F, s(0)=0\}$.

Theorem 3.1. *Let $\phi : E \to F$ be a mapping for which there exists a function $\Theta : E^p \to F$ with condition*

$$lim_{r\to\infty}V\left(\Theta(\eta^r t_1,\eta^r t_2,\cdots,\eta^r t_p),\eta^r\zeta\right) = 1 \quad \forall t_1,t_2,\cdots,t_p \in E,\ \zeta > 0 \tag{3.1}$$

and fulfilling the inequality

$$W(D\phi(t_1,t_2,\cdots,t_p),\zeta) \geq V(\Theta(t_1,t_2,\cdots,t_p),\zeta) \quad \forall t_1,t_2,\cdots,t_p \in E,\ \zeta > 0. \tag{3.2}$$

If there exist $L = L[i]$ such that the function $t \to \gamma(t) = \frac{1}{(p-1)}\Theta\left(0,\frac{t}{2},0,\cdots,0\right)$ has the property

$$V\left(L\frac{1}{\eta_a}\gamma(\eta_a t),\zeta\right) = V\left(\gamma(t),\zeta\right) \quad \forall t \in E,\ \zeta > 0, \tag{3.3}$$

then there exist unique additive function $A_1 : E \to F$ fulfilling the functional equation (1.2) and

$$W(\phi(t)-A_1(t),\zeta) \geq V\left(\frac{L^{1-a}}{1-L}\gamma(t),\zeta\right) \quad \forall t \in E,\ \zeta > 0.$$

Proof. Let d be a general metric on χ such that

$$d(m,n) = inf\left\{r \in (0,\infty)|W(m(t)-n(t),\zeta) \geq V(\gamma(t),r\zeta), t \in E, \zeta > 0\right\}.$$

It is easy to view that (χ,d) is complete. Define $T : \chi \to \chi$ by $Tm(t) = \frac{1}{\eta_a}m(\eta_a t)$ for all $t \in E$, for $m,n \in \chi$, we have

$$\begin{aligned} d(m,n) = r &\Rightarrow W\left(m(t)-n(t),\zeta\right) \geq V(\gamma(t),r\zeta), \\ &\Rightarrow W\left(\frac{m(\eta_a t)}{\eta_a}-\frac{n(\eta_a t)}{\eta_a},\zeta\right) \geq V(\gamma(\eta_a t),r\eta_a\zeta), \\ &\Rightarrow W(Tm(t)-Tn(t),\zeta) \geq V(\gamma(\eta_a t),r\eta_a\zeta), \\ &\Rightarrow W(Tm(t)-Tn(t),\zeta) \geq V(\gamma(t),rL\zeta), \\ &\Rightarrow d(Tm(t),Tn(t)) \geq rL, \\ &\Rightarrow d(Tm,Tn,\zeta) \geq Ld(m,n). \end{aligned} \tag{3.4}$$

for all $m, n \in \chi$. Therefore, T is strictly contractive mapping on χ with Lipschitz constant L, switching $(t_1, t_2, t_3, \cdots, t_p)$ by $(0, t, 0, \cdots, 0)$ in (3.2), we have

$$W\,(p-1)\phi(3t) - 3(p-1)\phi(t), \zeta) \geq V\,(\Theta(0, t, 0, \cdots, 0), \zeta) \quad \forall t \in E,\ \zeta > 0. \tag{3.5}$$

Using (N_3) in (3.5), we reach

$$W\left(\frac{\phi(3t)}{3} - \phi(t), \zeta\right) \geq V\left(\frac{\Theta(0, t, 0, \cdots, 0)}{3(p-1)}, \zeta\right) \quad \forall t \in E,\ \zeta > 0 \tag{3.6}$$

with the help of (3.3) when $a = 0$, it follows from (3.6) that

$$\Rightarrow W\left(\frac{\phi(3t)}{3} - \phi(t), \zeta\right) \geq V(L\gamma(t), \zeta)$$

$$\Rightarrow d(T\phi, \phi) \geq L = L^1 = L^{1-a}. \tag{3.7}$$

Replacing t by $\frac{t}{3}$ in (3.5), we get

$$W\left(\phi(t) - 3\phi\left(\frac{t}{3}\right), \zeta\right) \geq V\left(\frac{1}{(p-1)}\Theta\left(0, \frac{t}{3}, 0, \cdots, 0\right), \zeta\right) \quad \forall t \in E,\ \zeta > 0,$$

when $a = 1$, it follows from (3.7), we reach

$$\Rightarrow W\left(\phi(t) - 3\phi\left(\frac{t}{3}\right), \zeta\right) \geq V(\gamma(t), \zeta)$$

$$\Rightarrow T(\phi, T\phi) \leq 1 = L^0 = L^{1-a}. \tag{3.8}$$

Then from (3.7) and (3.8), we can conclude

$$\Rightarrow T(\phi, T\phi) \leq L^{1-a} < \infty.$$

Now, from the Theorem 1.1 in both cases, it follows that there exists a fixed point A_1 of T in χ such that

$$A_1(t) = W - \lim_{r \to \infty} \frac{\phi(\eta^r t)}{\eta^r}$$

for all $t \in E$ and $\zeta > 0$. Replacing $(t_1, t_2, \cdots, t_p)$ by $(\eta_a^r t_1, \eta_a^r t_2, \cdots, \eta_a^r t_p)$ in (3.2), we arrive

$$W\left(\frac{1}{\eta_a^r} D\phi(\eta_a^r t_1, \eta_a^r t_2, \cdots, \eta_a^r t_p), \zeta\right) \geq V(\Theta(\eta_a^r t_1, \eta_a^r t_2, \cdots, \eta_a^r t_p), \eta_a^r \zeta)$$

for all $\zeta > 0$ and all $t_1, t_2, \cdots, t_p \in E$. By proceeding the same procedure of the Theorem 2.1 , we can prove the function $A_1 : E \to F$ is additive and it fulfils (1.2). By a fixed point alternative, since A_1 is unique fixed point of T in the set

$$\Delta = \{\phi \in \chi / d(\phi, A_1) < \infty\}.$$

So that, A_1 is a unique function such that

$$W(\phi(t) - A_1(t), \zeta) \geq V(\gamma(t), r\zeta) \quad \forall t \in E,\ \zeta > 0.$$

Again using Theorem 1.1, we obtain

$$d(\phi, A_1) \leq \frac{1}{1-L} d(\phi, T\phi)$$

$$\Rightarrow d(\phi, A_1) \leq \frac{L^{1-a}}{1-L}$$

$$\Rightarrow W(\phi(t) - A_1(t), \zeta) \geq V\left(\gamma(t)\frac{L^{1-a}}{1-L}, \zeta\right) \quad \forall t \in E,\ \zeta > 0.$$

Hence the proof. □

Corollary 3.2. *Suppose a function $\phi : E \to F$ fulfils the inequality*

$$W(D\phi(t_1, t_2, \cdots, t_p), \zeta) \geq \begin{cases} V(\lambda, \zeta), \\ V(\lambda \sum_{a=1}^{p} ||t_a||^{\psi}, \zeta), \\ V(\lambda(\sum_{a=1}^{p} ||t_a||^{p\psi} + \prod_{a=1}^{p} ||t_a||^{\psi}), \zeta), \end{cases}$$

for all $t_1, t_2, \cdots, t_p \in E$ and $\zeta > 0$, where λ, ψ are constants with $\lambda > 0$. Then there exists a unique additive mapping $A_1 : E \to F$ such that

$$W(\phi(t) - A_1(t), \zeta) \geq \begin{cases} V(\lambda, \frac{\zeta(p-1)}{|2|}, \\ V\left(\lambda||t||^{\psi}, \zeta(p-1) \mid 3 - 3^{\psi} \mid\right); & \psi \neq 1, \\ V\left(\lambda||t||^{p\psi}, \zeta(p-1) \mid 3 - 3^{p\psi} \mid\right); & \psi \neq \frac{1}{p}. \end{cases}$$

for all $t \in E$ and $\zeta > 0$.

Proof. Setting

$$\Theta(t_1, t_2, t_3, \cdots, t_p) \leq \begin{cases} \lambda, \\ \lambda(\sum_{a=1}^{p} ||t_a||^{\psi}), \\ \lambda(\prod_{a=1}^{p} ||t_a||^{\psi} + \sum_{a=1}^{p} ||t_a||^{p\psi}). \end{cases}$$

for all $t_1, t_2, \cdots, t_p \in E$. Then

$$V\left(\Theta\left(\eta_a^r t_1, \eta_a^r t_2, \cdots, \eta_a^r t_p\right), \eta_a^r \zeta\right) = \begin{cases} V(\lambda, \eta_a^r \zeta), \\ V\left(\lambda \sum_{a=1}^{p} ||t_a||^{\psi}, \eta_a^{(1-\psi)r} \zeta\right), \\ V\left(\lambda(\sum_{a=1}^{p} ||t_a||^{p\psi} + \prod_{a=1}^{p} ||t_a||^{\psi}), \eta_a^{(1-p\psi)r} \zeta\right). \end{cases}$$

$$= \begin{cases} \to \quad 1 \quad as \quad r \to \infty, \\ \to \quad 1 \quad as \quad r \to \infty, \\ \to \quad 1 \quad as \quad r \to \infty. \end{cases}$$

Thus, (2.1) is holds. But we get

$$\gamma(t) = \frac{1}{(p-1)} \Theta\left(0, \frac{t}{3}, 0, \cdots, 0\right)$$

has the property

$$V\left(L \frac{1}{\eta_a} \gamma(\eta_a t), \zeta\right) \geq V(\gamma(t), \zeta) \quad \forall t \in E, \ \zeta > 0.$$

Hence,

$$V(\gamma(t), \zeta) = V\left(\Theta\left(0, \frac{t}{3}, 0, \cdots, 0\right), (p-1)\zeta\right)$$

$$= \begin{cases} V(\lambda, \zeta(p-1)), \\ V\left(\frac{1}{3^{\psi}} \lambda ||t||^{\psi}, \zeta(p-1)\right), \\ V\left(\frac{1}{3^{p\psi}} \lambda ||t||^{p\psi}, \zeta(p-1)\right). \end{cases}$$

Now,

$$V\left(\frac{1}{\eta_a} \gamma(\eta_a t), \zeta\right) = \begin{cases} V\left(\frac{\lambda}{\eta_a}, \zeta(p-1)\right), \\ V\left(\frac{\lambda}{\eta_a} \left(\frac{1}{3^{\psi}}\right) ||\eta_a t||^{\psi}, \zeta(p-1)\right), \\ V\left(\frac{\lambda}{\eta_a} \left(\frac{1}{3^{p\psi}}\right) ||\eta_a t||^{p\psi}, \zeta(p-1)\right). \end{cases}$$

$$= \begin{cases} V(\eta_a^{-1}\gamma(t),\zeta), \\ V(\eta_a^{\psi-1}\gamma(t),\zeta), \\ V(\eta_a^{p\psi-1}\gamma(t),\zeta). \end{cases}$$

Now, from the following cases for the conditions (i) and (ii).

Case(i): $L = 3^{-1} \quad for \quad \psi = 0 \quad if \quad a = 0.$

$$W(\phi(t) - A_1(t),\zeta) \geq V\left(\tfrac{L^{1-a}}{1-L}\gamma(t),\zeta\right) \geq V\left(\tfrac{3^{-1}}{1-3^{-1}}\tfrac{\lambda}{(p-1)},\zeta\right) \geq V\left(\lambda, 2\zeta(p-1)\right).$$

Case(ii): $L = \left(\frac{1}{3}\right)^{-1} \quad for \quad \psi = 0 \quad if \quad a = 1.$

$$W(\phi(t) - A_1(t),\zeta) \geq V\left(\tfrac{L^{1-a}}{1-L}\gamma(t),\zeta\right) \geq V\left(\tfrac{1}{1-\left(\frac{1}{3}\right)^{-1}}\tfrac{\lambda}{(p-1)},\zeta\right) \geq V\left(\lambda, -2\zeta(p-1)\right).$$

Case(iii): $L = (3)^{\psi-1} \quad for \quad \psi < 1 \quad if \quad a = 0.$

$$\begin{aligned} W(\phi(t) - A_1(t),\zeta) \geq V\left(\frac{L^{1-a}}{1-L}\gamma(t),\zeta\right) &\geq V\left(\frac{3^{\psi-1}}{1-3^{\psi-1}}\frac{\lambda||t||^{\psi}}{(p-1)3^{\psi}},\zeta\right) \\ &\geq V\left(\lambda||t||^{\psi},\zeta(p-1)(3-3^{\psi})\right). \end{aligned}$$

Case(iv): $L = (3)^{1-\psi} \quad for \quad \psi > 1 \quad if \quad a = 1.$

$$\begin{aligned} W(\phi(t) - A_1(t),\zeta) \geq V\left(\frac{L^{1-a}}{1-L}\gamma(t),\zeta\right) &\geq V\left(\frac{3^{1-\psi}}{1-3^{1-\psi}}\frac{\lambda||t||^{\psi}}{(p-1)3^{\psi}},\zeta\right) \\ &\geq V\left(\lambda||t||^{\psi},\zeta(p-1)(3^{\psi}-3)\right). \end{aligned}$$

Case(v): $L = (3)^{p\psi-1} \quad for \quad \psi < \frac{1}{p} \quad if \quad a = 0.$

$$\begin{aligned} W(\phi(t) - A_1(t),\zeta) \geq V\left(\frac{L^{1-a}}{1-L}\gamma(t),\zeta\right) &\geq V\left(\frac{3^{p\psi-1}}{1-3^{p\psi-1}}\frac{\lambda||t||^{p\psi}}{(p-1)3^{p\psi}},\zeta\right) \\ &\geq V\left(\lambda||t||^{p\psi},\zeta(p-1)(3-3^{p\psi})\right). \end{aligned}$$

Case(vi): $L = (3)^{1-p\psi} \quad for \quad \psi > \frac{1}{p} \quad if \quad a = 1.$

$$\begin{aligned} W(\phi(t) - A_1(t),\zeta) \geq V\left(\frac{L^{1-a}}{1-L}\gamma(t),\zeta\right) &\geq V\left(\frac{3^{1-p\psi}}{1-3^{1-p\psi}}\frac{\lambda||t||^{p\psi}}{(p-1)3^{p\psi}},\zeta\right) \\ &\geq V\left(\lambda||t||^{p\psi},\zeta(p-1)(3^{p\psi}-3)\right). \end{aligned}$$

Hence the proof is completed. □

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