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ON PROPERTIES OF HESITANT FUZZY IDEALS IN SEMIGROUPS

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ABSTRACT. In this paper, we study some properties of hesitant fuzzy ideals and hesitant fuzzy bi-ideals in a semigroup and discuss their characterizations. Also we introduce hesitant fuzzy interior ideals in a semigroup and studied their properties. It is proved that in a semigroup a hesitant fuzzy ideal is a hesitant fuzzy interior ideal but the converse is not true. Moreover we prove that in regular and in intra-regular semigroups the hesitant fuzzy ideals and the hesitant fuzzy interior ideals coincide.

1. INTRODUCTION

Zadeh [5] introduced the concept of a fuzzy set. Given a set S, a fuzzy subset of S (or a fuzzy set in S) is described as an arbitrary mapping f : $S \rightarrow$ [0,1] where [0,1] is the usual interval of real numbers. Fuzzy groups have been first studied by Rosenfeld [2]. He gave the definition of fuzzy subgroupoid and the fuzzy left(right,two-sided) ideal. Kuroki [7, 8] has first studied the fuzzy sets in semigroups . Lius [15] gave several further fuzzy notions such as fuzzy bi-deals and fuzzy interior ideals.

Torra [13] introduced the concept of hesitant fuzzy set as a useful generalisation of the fuzzy set that is designed for situations in which it is difficult to determine the membership of an element to a set owing to hesitancy between a few different values. The hesitant fuzzy set permits the membership degree of an element to a set to be represented by a set of possible values between 0 and 1 [13, 14]. Jun and Song [16] applied the notion of hesitant fuzzy sets to MTL-algebras and EQ-algebras. Ali et al. [1] applied the notion of hesitant fuzzy sets to AG-groupoids. Jun et al. [17] applied the notion of hesitant fuzzy sets to semigroups and hesitant fuzzy soft sets to subalgebras and BCK/BCI-algebras. They introduced the notion of hesitant fuzzy semigroups and hesitant fuzzy left (resp., right) ideals and reviewed several properties. Our main goal in this paper is to introduce and study the new sort of hesitant fuzzy ideals in semigroup.

Throughout this paper, unless stated otherwise, S stands for semigroup.

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2. Preliminaries

For any two subsets X and Y of a semigroup S, the multiplication of X and Y is defined as follows:

$$XY = \{xy \in S \mid x \in X \text{ and } y \in Y\}$$

Definition 2.1. A nonempty subset A of S is called a subsemigroup of S if $AA \subseteq A$, that is, $xy \in A$ for all $x, y \in A$.

Definition 2.2. A nonempty subset A of S is called a left (resp., right) ideal of S if $SA \subseteq A$ (resp., $AS \subseteq A$), that is, $xa \in A$ (resp., $ax \in A$) for all $x \in$ S and $a \in A$.

Definition 2.3. A nonempty subset A of S is called a two-sided ideal of S if it is both a left and a right ideal of S.

Definition 2.4. A nonempty subset A of S is called a generalized bi-ideal of S if $ASA \subseteq A$.

Definition 2.5. A nonempty subset A of S is called a bi-ideal of S if it is both a semigroup and a generalized bi-ideal of S.

We now discuss the basic notions on hesitant fuzzy sets. Let S be a reference set. Then we define hesitant fuzzy set on S in terms of a function H that when applied to S returns a subset of [0, 1].

Let H be any hesitant fuzzy set on S and $a, b, c \in$ S, we use the notations $H_a := H(a)$, $H_a^b := H(a) \cap H(b)$, $H_a^b[c] := H(a) \cap H(b) \cap H(c)$, $[H_A]_a := H_A(a)$ and $[H_A]_a^b := H_A(a) \cap H_A(b)$. It is clear that $H_a^b := H_b^a$ and $H_a := H_b \Leftrightarrow H_a \subseteq H_b$, $H_b \subseteq H_a$ for all $a, b \in S$.

Let H and G be any two hesitant fuzzy sets on S, we define $H \sqsubseteq G$ if $H_a \subseteq G_a$ for all $a \in S$.

For any hesitant fuzzy sets H and G in S. The hesitant fuzzy product of H and G is defined to be the hesitant fuzzy set $H\tilde{o}G$ on S as follows:

$$(H\tilde{o}G)_a = \begin{cases} \bigcup\limits_{a=bc} (H_b \cap G_c)\; if\; \exists\, b, c \in S \text{ such that } a = bc \\ \phi\; otherwise \end{cases} \tag{1}$$

For any two hesitant fuzzy sets H and G on S, let$\mathcal{P}([0,1])$ denotes the set of all subsets of [0,1], the hesitant union $H \sqcup G$ of H and G is defined to be hesitant fuzzy set on S as follows:

$$H \sqcup G : S \to \mathcal{P}([0,1]), a \longmapsto H_a \cup G_a$$

and the hesitant intersection $H \sqcap G$ of H and G is defined to be hesitant fuzzy set on S as follows:

$$H \sqcap G : S \to \mathcal{P}([0,1]), a \longmapsto H_a \cap G_a.$$

For a non empty subset A of a semigroup S, then $H_A : S \to \mathcal{P}([0,1])$ where the function

$$H_A(x) = \begin{cases} [0,1]\; if\; x \in A \\ \phi\; if\; x \notin A \end{cases}$$

is called the hesitant characteristic function of A.

Definition 2.6. A hesitant fuzzy set H on S is called a hesitant fuzzy semigroup on S if it satisfies :

$$(\forall\, x, y \in S)(H_x^y \subseteq H_{xy}).$$

Definition 2.7. A hesitant fuzzy set H on S is called a hesitant fuzzy left (resp., right) ideal on S if it satisfies :

$$(\forall\, x, y \in S)(H_y \subseteq H_{xy}(resp., H_x \subseteq H_{xy})).$$

If a hesitant fuzzy set H on S is both a hesitant fuzzy left ideal and a hesitant fuzzy right ideal on S, we say that H is a hesitant fuzzy two-sided ideal on S.

Example 2.1. Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table .

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let H be a hesitant fuzzy subset of S such that

$$H_a = [0, 1];\ H_b = [0.2, 0.8];\ H_c = \phi;\ H_d = [0.5, 0.6]$$

Clearly

$$H_{(xy)} = \begin{cases} H_b\ if\ (x, y) := \begin{cases} (c, c) \\ (d, d) \\ (d, c) \end{cases} \\ H_a\ otherwise \end{cases} \tag{2}$$

for every x and y of S. Hence H is a hesitant fuzzy ideal of S.

Obviously, every hesitant fuzzy left (resp., right) ideal on S is a hesitant fuzzy semigroup on S. But the converse is not true in general.

Definition 2.8. A hesitant fuzzy subsemigroup H on S is called a hesitant fuzzy bi-ideal on S if it satisfies:

$$(\forall\, x, y, z \in S)(H_{xyz} \supseteq H_x^z).$$

Definition 2.9. A hesitant fuzzy set H on S is called a hesitant fuzzy generalized bi-ideal on S if it satisfies:

$$(\forall\, x, y, z \in S)(H_{xyz} \supseteq H_x^z).$$

Lemma 2.1. *Every hesitant fuzzy left ideal (hesitant fuzzy right ideal & hesitant fuzzy ideal) is a hesitant fuzzy bi-ideal of S.*

Proof. Let H be any hesitant fuzzy left ideal of S and x, y and z any elements of S. Then

$$H_{xyz} = H_{(xy)z} \supseteq H_z \supseteq H_x^z.$$

Hence H is a hesitant fuzzy bi-ideal of S. □

Similarly we can see the other cases hold.

3. Regular semigroups

Definition 3.1. A semigroup S is called regular if, for each element a of S, there exists an element x in S such that

$$a = axa.$$

Definition 3.2. A semigroup S is called hesitant fuzzy left (right) duo if every hesitant fuzzy left (right) ideal of S is a hesitant fuzzy ideal of S. A semigroup S is called hesitant fuzzy duo if it is both hesitant fuzzy left and hesitant fuzzy right duo.

Definition 3.3. A semigroup S is called right(left) zero if

$$xy = y(xy = x) \quad \forall\, x, y \in S.$$

Lemma 3.1. *Let A be a non-empty subset of a semigroup S and H_A the hesitant characteristic function of A. Then the following hold:*

(1) A is a subsemigroup of S if and only if H_A is a hesitant fuzzy subsemigroup of S.

(2) A is a left(right, two-sided) ideal of S if and only if H_A is a hesitant fuzzy left(right, two-sided) ideal of S.

Proof. (1) Let A be a subsemigroup of S. For any $x, y \in S$. We prove that

$$[H_A]_{(xy)} \supseteq [H_A]_x^y. \tag{3}$$

If $x, y \in A$, then $xy \in A$. Therefore $[H_A]_x = [H_A]_y = [H_A]_{(xy)} = [0,1]$. Hence $[H_A]_{(xy)} = [H_A]_x^y$ and 3 is satisfied. If $x \notin A$ or $y \notin A$, then $[H_A]_x = \phi$ or $[H_A]_y = \phi$ implies $[H_A]_x^y = \phi$. Now if $xy \in A$ implies $[H_A]_{(xy)} = [0,1]$. Hence $[H_A]_{(xy)} \supseteq [H_A]_x^y$ and 3 is satisfied. Now if $xy \notin A$ implies $[H_A]_{(xy)} = \phi = [H_A]_x^y$ and 3 is satisfied.

Conversely, assume that H_A is a hesitant fuzzy subsemigroup on S. Let $x, y \in A$. Then $[H_A]_x = [H_A]_y = [0,1]$. Since $[H_A]_{(xy)} \supseteq [H_A]_x^y = [0,1]$. Hence $[H_A]_{(xy)} = [0,1]$ and so $xy \in A$. Thus A is a subsemigroup of S.

(2) Let A be a left ideal of S. For any $x, y \in S$. We prove that

$$[H_A]_{(xy)} \supseteq [H_A]_y. \tag{4}$$

If $y \notin A$, then $[H_A]_y = \phi \subseteq [H_A]_{(xy)}$ and 4 is satisfied. If $y \in A, x \in S$, then $xy \in A$. Hence $[H_A]_{(xy)} = [0,1] = [H_A]_y$ and 4 is satisfied.

Conversely, Let H_A is a hesitant fuzzy left ideal on S. Let $x \in S$ and $y \in A$. Then $[H_A]_y = [0,1]$. Therefore, $[H_A]_{(xy)} \supseteq [H_A]_y = [0,1]$. Hence $[H_A]_{(xy)} = [0,1]$ and so $xy \in A$.

Thus, A is a left ideal of S. □

Lemma 3.2. *Let A be a non-empty subset of a semigroup S and H_A the hesitant characteristic function of S. Then A is a bi-ideal of S if and only if H_A is a hesitant fuzzy bi-ideal of S.*

Proof. Let A be a bi-ideal of S. For any $x, y, z \in S$. We prove that

$$[H_A]_{(xyz)} \supseteq [H_A]_x^z. \tag{5}$$

If $x \in A$ and $z \in A$, then $[H_A]_x = [H_A]_z = [0,1]$. Since A is a bi-ideal of S, $xyz \in ASA \subseteq A$ implies $[H_A]_{(xyz)} = [0,1]$. Hence $[H_A]_{(xyz)} = [H_A]_x^z$ and 5 is satisfied. If If $x \notin A$ or $z \notin A$, then $[H_A]_x = \phi$ or $[H_A]_z = \phi$. Hence $[H_A]_x^z = \phi \subseteq [H_A]_{(xyz)}$ and 5 is satisfied. It follows from Lemma 3.1, H_A is a hesitant fuzzy subsemigroup of S. Hence H_A is a hesitant fuzzy bi-ideal of S.

Conversly, assume that H_A is a hesitant fuzzy bi-ideal of S. For any $x, y, z \in S$. Let $xyz \in ASA$ suchthat $x, z \in A, y \in S$ then $[H_A]_x = [H_A]_z = [0,1]$. Since H_A is a hesitant fuzzy bi-ideal of S, we have $[H_A]_{(xyz)} \supseteq [H_A]_x^z = [0,1]$. Hence $[H_A]_{(xyz)} = [0,1]$ implies $xyz \in A$. Thus $ASA \subseteq A$. It follows from Lemma 3.1, A is a subsemigroup of S. Hence A is a bi-ideal of S. □

Theorem 3.3. *The following conditions are equivalent for a regular semigroup S:*

(1) S is left duo.

(2) S is a hesitant fuzzy left duo.

Proof. (1) $\Rightarrow$ (2). Let H be any hesitant fuzzy left ideal of S and a, b be any elements of S. Then, since the left ideal Sa is a two-sided ideal of S, and S is regular, we have

$ab \in (aSa)b \subseteq (Sa)S \subseteq Sa.$
This implies that there exists an element x in S such that
$ab = xa.$
Then, since H is a hesitant fuzzy left ideal of S, we have
$H_{ab} = H_{xa} \supseteq H_a.$
This means that H is a hesitant fuzzy right ideal of S, and so H is a hesitant fuzzy two-sided ideal of S. Thus we obtain that S is hesitant fuzzy left duo.

(2) ⇒ (1). Suppose that S is a hesitant fuzzy left duo. Let A be any left ideal of S. Then it follows from Lemma 3.1 that the hesitant characteristic function H_A of A is a hesitant fuzzy left ideal of S. Thus by the assumption H_A is a hesitant fuzzy ideal of S. Since A is non-empty, it follows from Lemma 3.1 that A is an ideal of S. Hence S is left duo. □

The right dual of Theorem 3.3 read as follows:

Theorem 3.4. *The following conditions are equivalent for a regular semigroup S:*
(1) S is right duo.
(2) S is a hesitant fuzzy right duo.

The following theorem is the immediate consequence of Theorem 3.3 and Theorem 3.4.

Theorem 3.5. *The following conditions are equivalent for a regular semigroup S:*
(i) S is duo.
(ii) S is a hesitant fuzzy duo.

Theorem 3.6. *The following conditions are equivalent for a regular semigroup S:*
(1) Every bi-ideal of S is a right ideal of S.
(2) Every hesitant fuzzy bi-ideal of S is a hesitant fuzzy right ideal of S.

Proof. (1) ⇒ (2). Assume every bi-ideal of S is a right ideal of S. Let H be any hesitant fuzzy bi-ideal of S, and a, b any elements of S. Then, aSa is a bi-ideal of S. Thus by the assumption aSa is a right ideal of S. Since S is regular, we have
$ab \in (aSa)S \subseteq aSa.$
This implies that there exist an element x in S such that
$ab = axa.$
Since H is a hesitant fuzzy bi-ideal of S, we have
$H_{ab} = H_{(axa)} \supseteq H_a^a = H_a.$
This means that H is a fuzzy right ideal of S.

(2) ⇒ (1). Suppose that every hesitant fuzzy bi-ideal of S is a hesitant fuzzy right ideal of S. Let A be any bi-ideal of S. Then it follows from Lemma 3.2 that the hesitant characteristic function H_A of A is a hesitant fuzzy bi-ideal of S. Thus by the assumption H_A is a hesitant fuzzy right ideal of S. Since A is non-empty, it follows from Lemma 3.2 that A is right ideal of S. □

The left dual of Theorem 3.6 is proved in an analogous way:

Theorem 3.7. *The following conditions are equivalent for a regular semigroup S:*
(1) Every bi-ideal of S is a left ideal of S.
(2) Every hesitant fuzzy bi-ideal of S is a hesitant fuzzy left ideal of S.

The following theorem is the immediate consequence of Theorem 3.6 and Theorem 3.7.

Theorem 3.8. *The following conditions are equivalent for a regular semigroup S:*
(1) Every bi-ideal of S is a two-sided ideal of S.
(2) Every hesitant fuzzy bi-ideal of S is a hesitant fuzzy two-sided ideal of S.

Theorem 3.9. *The following conditions are equivalent for a regular semigroup S:*
(1) The set of all idempotent elements E of S forms a left zero subsemigroup of S.
(2) For every hesitant fuzzy left ideal H of S, $H_e = H_f$ holds for all idempotent elements e and f
of S.

Proof. (1) $\Rightarrow$ (2). Suppose that the set E of all idempotents of S forms a left zero sub semigroup of S. Let e and f be any elements of E and H be any hesitant fuzzy left ideal of S.
Since $ef = e$ and $fe = f$, we have

$$H_e = H_{ef} \supseteq H_f = H_{fe} \supseteq H_e$$

and so $H_e = H_f$.

(2) $\Rightarrow$ (1). Suppose that for every hesitant fuzzy left ideal H of S, $H_e = H_f$ for all idempotent elements e and f of S. We denote by $L[e]$ the principal left ideal of a semigroup S generated by e in S, that is, $L[e] = e \cup Se$. Since S is regular, E is non-empty. Then it follows from Lemma 3.1 that the hesitant characteristic function $H_{L[e]}$ of the left ideal $L[e]$ of S is a hesitant fuzzy left ideal of S. Since $e \in L[e]$ implies $H_{L[e]}(f) = H_{L[e]}(e) = [0, 1]$ and so $f \in L[e] = Se$. Then for some $x \in S$, we have $f = xe = xee = fe$. Hence E is a left zero subsemigroup of S.

□

Corollary 3.10. *For an idempotent semigroup S, if S is left zero then for every hesitant fuzzy left ideal H of S, we have $H_e = H_f$ holds for all e and $f \in S$.*

The right dual of Theorem 3.9 reads as follows:

Theorem 3.11. *For a regular semigroup S if the set of all idempotent elements E of S forms a right zero sub semigroup of S. Then every hesitant fuzzy right ideal H of S, $H_e = H_f$ holds for all idempotent elements e and f of S.*

Corollary 3.12. *For an idempotent semigroup S, if S is right zero then for every hesitant fuzzy right ideal H of S, we have $H_e = H_f$ holds for all $e, f \in S$.*

Theorem 3.13. *If S is a group, then every hesitant fuzzy bi-ideal of S is a constant function.*

Proof. Let S be a group with identity e and let H be a hesitant fuzzy bi-ideal on S. For any $a \in S$, we have

$$H_a = H_{eae} \supseteq H_e^e = H_e = H_{ee} = H_{(aa^{-1})(a^{-1}a)} = H_{(a(a^{-1}a^{-1})a)} \supseteq H_a^a = H_a$$

and so $H_a = H_e$.
Therefore H is a constant function. □

Corollary 3.14. *If S is a group, then for every hesitant fuzzy bi-ideal H of S, $H_e = H_f$ holds for all idempotent elements $e, f \in S$.*

Proof. By Theorem 3.13 H is a constant function. Therefore for any idempotents e and f of S, we have $H_e = H_f$. □

4. INTRA-REGULAR SEMIGROUPS

Definition 4.1. A semigroup S is called intra-regular if, for each element a of S, there exist elements x and y in S such that

$$a = xa^2y.$$

Theorem 4.1. *Let S be an intra-regular semigroup. Then for every hesitant fuzzy ideal H of S $H_a = H_{a^2}$ and $H_{ab} = H_{ba}$ $\forall\, a, b \in S$.*

Proof. Let S be an intra regular semigroup and a, b be any element of S. Let H be hesitant fuzzy ideal of S. Then, since S is intra regular, there exist elements x and y in S such that

$$a = xa^2y.$$

Then, since H is a hesitant fuzzy ideal of S, we have

$$H_a = H_{xa^2y} \supseteq H_{xa^2} \supseteq H_{a^2} \supseteq H_a$$

and so we have, $H_a = H_{a^2}$.
Since $ab \in S$, we have

$$H_{ab} = H_{(ab)^2} = H_{(a(ba)b)} \supseteq H_{ba}.$$

Again

$$H_{ba} = H_{(ba)^2} = H_{(b(ab)a)} \supseteq H_{ab}.$$

and so we have

$$H_{ab} = H_{ba}.$$ □

5. Completely regular semigroups

Definition 5.1. A semigroup S is called completely regular if, for each element a of S, there exists an element x in S such that

$$a = axa \quad and \quad ax = xa.$$

Definition 5.2. A semigroup S is called left(right) regular if, for each element a of S, there exists an element x in S such that

$$a = xa^2 \ (a = a^2x).$$

Theorem 5.1. *Let S be left regular semigroup. Then for every hesitant fuzzy left ideal H of S,*

$H_a = H_{a^2}$ holds for all $a \in S$.

Proof. Let H be any hesitant fuzzy ideal of S and a be any element of S. Then, since S is left regular, there exist an element x in S such that

$$a = xa^2.$$

Then we have

$$H_{a^2} \supseteq H_a \supseteq H_{x(a^2)} \supseteq H_{a^2}$$

and so we have

$$H_a = H_{a^2}.$$ □

The right dual of Theorem 5.1 reads as follows:

Theorem 5.2. *Let S be right regular semigroup. Then for every hesitant fuzzy right ideal H of S, $H_a = H_{a^2}$ holds for all $a \in S$.*

Theorem 5.3. *Let S be completely regular semigroup. Then for every hesitant fuzzy bi-ideal H of S, $H_a = H_{a^2}$ holds for all $a \in S$.*

Proof. Let S be a completely regular semigroup. Let H be any hesitant fuzzy bi-ideal of S and a be any element of S. Then there exists an element x in S such that

$$a = a^2xa^2.$$

Since H is a hesitant fuzzy bi-ideal of S, we have

$$H_a = H_{(a^2xa^2)} \supseteq H_{a^2}^{a^2} = H_{a^2} \supseteq H_a^a = H_a$$

and so we have

$$H_a = H_{a^2}.$$ □

6. Hesitant fuzzy interior ideals

Definition 6.1. A hesitant fuzzy set H on S is called a hesitant fuzzy interior ideal on S if it satisfies:

$$(\forall x, y, z \in S)(H_{xyz} \supseteq H_y).$$

It is easy to see that, every hesitant fuzzy two-sided ideal of a semigroup S is a hesitant fuzzy interior ideal of S. The following example show that the converse of this property does not hold in general.

Example 6.1. Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table .

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let H be a hesitant fuzzy subset of S such that $H_a = [0, 1];\ H_b = \phi;\ H_c = [0.2, 0.4];\ H_d = \phi$

$$H_{(xyz)} = H_a = [0, 1] \supseteq H_y$$

for every x, y, and z of S.
Then H is a hesitant fuzzy interior ideal of S. But it is not a hesitant fuzzy ideal of S since,

$$H_{(dc)} = H_b = \phi \not\supseteq [0.2, 0.4] = H_c.$$

Lemma 6.1. *Let A be a non-empty subset of a semigroup S and H_A the hesitant characteristic function of S. Then A is interior ideal of S if and only if H_A is a hesitant fuzzy interior ideal of S.*

Proof. Let A be a interior ideal of S. For any $x, y, z \in S$. We prove that

$$[H_A]_{(xyz)} \supseteq [H_A]_y. \tag{6}$$

If $y \in A$, then $xyz \in SAS \subseteq A$. This implies that $[H_A]_{(xyz)} = [0, 1] = [H_A]_y$ and 6 is satisfied. If $y \notin A$, then $[H_A]_y = \phi \subseteq [H_A]_{(xyz)}$ and 6 is satisfied. It follows from Lemma 3.1, H_A is a hesitant fuzzy subsemigroup of S. Hence H_A is a hesitant fuzzy interior ideal of S.

Conversly, assume that H_A is a hesitant fuzzy interior ideal of S. For any $x, y, z \in S$. Let $xyz \in SAS$ such that $x, z \in S, y \in A$ then $[H_A]_{xyz} \supseteq [H_A]_y = [0, 1]$. Hence $[H_A]_{(xyz)} = [0, 1]$ implies $xyz \in A$. Thus $SAS \subseteq A$. It follows from Lemma 3.1, A is a subsemigroup of S. Hence A is an interior ideal of S. □

Lemma 6.2. *Let S be a semigroup and H a hesitant fuzzy ideal of S. Then H is a hesitant fuzzy interior ideal of S.*

Proof. Let H a hesitant fuzzy ideal of S. For any x, y, $z \in S$, we have

$$H_{xyz} = H_{(x(yz)} \supseteq H_{yz} \supseteq H_y.$$

Therefore H is a hesitant fuzzy interior ideal of S.

□

Theorem 6.3. *For a regular semigroup S and H be any hesitant fuzzy set in S the following conditions are equivalent.*

(1) H is a hesitant fuzzy ideal of S.

(2) H is a hesitant fuzzy interior ideal of S.

Proof. (1) ⇒ (2) by Lemma 6.2.Now we show (2) ⇒ (1). Let a, b be any elements of S. Then, since S is regular, there exist elements x and y of S such that $a = axa$ and $b = byb$. Then, since H is a hesitant fuzzy interior ideal of S, we have

$$H_{ab} = H_{(axa)b} = H_{((ax)ab)} \supseteq H_a$$

and

$$H_{ab} = H_{(a(byb)} = H_{((ab(yb))} \supseteq H_b.$$

This means that H is a hesitant fuzzy ideal of S. □

Theorem 6.4. *For any hesitant fuzzy set H of an intra regular semigroup the following conditions are equivalent.*

(1) H is a hesitant fuzzy ideal of S.

(2) H is a hesitant fuzzy interior ideal of S.

Proof. (1) ⇒ (2) by Lemma 6.2. Now we show (2) ⇒ (1). Let x and y be any elements of S . Then, since S is intra-regular, there exists elements m, n, p and q in S such that $x = mx^2n$ and $y = py^2q$.

Since H is a hesitant fuzzy interior ideal of S,

We have

$$H_{xy} = H_{((mx^2n)y)} = H_{((mx)x(ny))} = H_x$$

and

$$H_{xy} = H_{((x(py^2q))} = H_{((xp)y(yq))} = H_y.$$

This means that H is hesitant fuzzy ideal of S. □

Theorem 6.5. *Let S be an intra-regular semigroup. Then for every hesitant fuzzy interior ideal H of S, $H_a = H_{a^2}$ and $H_{ab} = H_{ba}$ $\forall\, a, b \in S$.*

Proof. Let H be hesitant fuzzy interior ideal of S and $a,\ b$ be any element of S.

Since S is intra regular semigroup, there exist elements x and y in S such that $a = xa^2y$.

Thus, $H_{a^2} \supseteq H_a = H_{xa^2y} \supseteq H_{a^2}$

and so we have, $H_a = H_{a^2}$.

Since $ab \in S$, we have

$$H_{ab} = H_{(ab)^2} = H_{(a(ba)b)} \supseteq H_{ba}.$$

Again

$$H_{ba} = H_{(ba)^2} = H_{(b(ab)a)} \supseteq H_{ab}.$$

and so we have

$H_{ab} = H_{ba}$. □

7. Conclusion

The findings and conclusion of the work carried out in this paper are following:
The concept of hesitant fuzzy ideals, hesitant fuzzy bi-ideals and hesitant fuzzy interior ideals in a semigroup and their related properties. We discussed that in a semigroup a hesitant fuzzy ideal is a hesitant fuzzy interior ideal but the converse is not true. Finally We show that in regular and in intra-regular semigroups the hesitant fuzzy ideals and the hesitant fuzzy interior ideals coincide.

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