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RICCI CURVATURE INEQUALITIES FOR WARPED PRODUCT SKEW CR-SUBMANIFOLDS IN COSYMPLECTIC SPACE FORMS

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ABSTRACT. The main objective of this paper is to achieve the Chen-Ricci inequality for skew CR-warped product submanifold isometrically immersed in a Cosymplectic space form in the expressions of the squared norm of mean curvature vector and warping functions. The equality cases is likewise discussed. However, in particular, we also derive Chen-Ricci inequality for CR-warped product submanifolds.

1. INTRODUCTION

A. Bejancu [1] presented and studied the idea of CR-submanifolds in 1981 as a generalization of holomorphic and totally real submanifolds. Further, Chen [6] investigated the geometry of CR-submanifolds of almost Hermitian manifolds in greater depth in order to gain a better understanding of their geometry. In 1990 Chen [7] introduced a generalized class of submanifolds namely slant submanifolds. Moreover, advances in geometry of CR-submanifolds and slant submanifolds stimulate various authors to search for the class of submanifolds which unifies the properties of all previously discussed submanifolds. In this context, N. Papaghuic [21]introduced the notion of Semi-slant submanifolds in the frame of almost Hermitian manifolds and showed that submanifolds belonging to this class enjoy many of the desired properties. Later, the contact variant of Semi-slant submanifolds was studied by Cabrerizo et al. [17]. Recently, B. Sahin [4] investigated another class of submanifolds in the setting of almost Hermitian manifolds and he called these submanifolds Hemi-slant submanifold. This class includes the CR-submanifolds and Slant submanifolds.

In 1990, Ronsse [16] started the study of skew CR-submanifolds in the setting of almost Hermitian manifolds. skew CR-submanifolds contain the class of CR-submanifolds, Semi-slant submanifolds and Hemi-slant submanifolds.

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The acknowledgment of warped product manifolds appeared after the methodology of Bishop and O'Neill [22] on the manifolds of negative curvature. Analyzing the way that a Riemannian product of manifolds can not have negative curvature, they build the model of warped product manifolds for the class of manifolds of negative (or non positive) curvature which is characterized as follows:

Let (S_1, g_1) and (S_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively and ψ be a positive differentiable function on S_1. If $a : S_1 \times S_2 \to S_1$ and $b : S_1 \times S_2 \to S_2$ are the projection maps given by $a(x, y) = x$ and $b(x, y) = y$ for every $(x, y) \in S_1 \times S_2$, then the *warped product manifold* is the product manifold $M = S_1 \times S_2$ endowed with the Riemannian structure such that

$$g(X, Y) = g_1(x_*X, x_*Y) + (\psi \circ x)^2 g_2(y_*X, y_*Y),$$

for all $X, Y \in TM$. The function ψ is called the *warping function* of the warped product manifold. If the warping function is constant, then the warped product is trivial i.e., simply Riemannian product. Further, if $X \in TS_1$ and $Z \in TS_2$, then from Lemma 7.3 of [22], we have the following well known result

$$\nabla_X Z = \nabla_Z X = (\frac{X\psi}{\psi})Z, \tag{1.1}$$

where ∇ is the Levi-civita connection on M.

Some common properties of warped product manifolds were concentrated in [22]. B. Y. Chen ([6], [8]) played out an outward investigation of warped product submanifolds in a Kaehler manifold. From that point forward, numerous geometers have investigated warped product manifolds in various settings like almost complex and almost contact manifolds and different existence results have been researched (see the survey article [12]). Recently, B. Sahin [5] contemplated skew CR-warped product submanifolds in the setting of Kaehler manifolds and got some essential outcomes. Further, these submanifolds were explored by Haidar and Thakur in the frame of Cosymplectic manifolds[23].

In 1999, Chen [9] discovered a relationship between Ricci curvature and squared mean curvature vector for a discretionary Riemannian manifold. On the line of Chen a series of articles have been appeared to formulate the relationship between Ricci curvature and squared mean curvature in the setting of some important structures on Riemannian manifolds (see [13], [2], [15], [19], [20]).

2. Preliminaries

A $(2n+1)-$dimensional $C^\infty-$manifold $\bar{M}$ is said to have an *almost contact structure* if on $\bar{M}$ there exist a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying [14]

$$\phi^2 = -I + \eta \otimes \xi, \ \phi\xi = 0, \ \eta \circ \phi = 0, \ \eta(\xi) = 1. \tag{2.1}$$

The manifold $\bar{M}$ with the structure (ϕ, ξ, η) is called *almost contact metric manifold*. There always exists a Riemannian metric g on an almost contact metric manifold $\bar{M}$, satisfying the following conditions

$$\eta(X) = g(X, \xi), \ g(\phi X, \phi U) = g(X, U) - \eta(X)\eta(U), \tag{2.2}$$

for all $X, U \in T\bar{M}$ where $T\bar{M}$ is the tangent bundle of $\bar{M}$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be *Cosymplectic manifold* if it satisfies the following tensorial equation [14]

$$(\bar{\nabla}_X \phi)Y = 0, \tag{2.3}$$

for any $X, Y \in T\bar{M}$, where $\bar{\nabla}$ denotes the Riemannian connection of the metric g. Moreover, for a Cosymplectic manifold

$$\bar{\nabla}_X \xi = 0. \tag{2.4}$$

A Cosymplectic manifold $\bar{M}$ is said to be a *Cosymplectic space form* [14] if it has constant ϕ-holomorphic sectional curvature c and is denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of Cosymplectic space form $\bar{M}(c)$ is given by

$$\begin{aligned}\bar{R}(X,Y)Z =& \frac{c}{4}[\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) - g(\phi X,Z)g(\phi Y,W)\}\\ & g(\phi X,W)g(\phi Y,Z) + 2g(\phi X,Y)g(\phi Z,W) - \eta(Z)\eta(Y)g(X,W)\\ & + \eta(Z)\eta(X)g(Y,W) + \eta(Z)\eta(W)g(Y,Z) + \eta(Z)\eta(Y)g(X,Z),\end{aligned} \tag{2.5}$$

for any vector fields X, Y, Z, W on $\bar{M}$.

Let M be an $n-$dimensional Riemannian manifold isometrically immersed in a $m-$dimensional Riemannian manifold $\bar{M}$. Then the Gauss and Weingarten formulas are $\nabla_X Y = \nabla_X Y + h(X,Y)$ and $\nabla_X \xi = -A_\xi X + \nabla^\perp_X \xi$ respectively, for all $X, Y \in TM$ and $\xi \in T^\perp M$. Where ∇ is the induced Levi-civita connection on M, ξ is a vector field normal to M, h is the second fundamental form of M, $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$ and A_ξ is the shape operator of the second fundamental form. The second fundamental form h and the shape operator are associated by the following formula

$$g(h(X,Y),\xi) = g(A_\xi X, Y). \tag{2.6}$$

The equation of Gauss is given by

$$R(X,Y,Z,W) = \bar{R}(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \tag{2.7}$$

for all $X, Y, Z, W \in TM$. Where, R and $\bar{R}$ are the curvature tensors of M and $\bar{M}$ respectively.

For any $X \in TM$ and $N \in T^\perp M$, JX and JN can be decomposed as follows

$$JX = PX + FX \tag{2.8}$$

and

$$JN = tN + fN, \tag{2.9}$$

where PX (resp. tN) is the tangential and FX (resp. fN) is the normal component of JX (resp. JN).

It is evident that $g(\phi X, Y) = g(PX, Y)$ for any $X, Y \in T_xM$, this implies that $g(PX,Y) + g(X,PY) = 0$. Thus, P^2 is a symmetric operator on the tangent space T_xM, for all $x \in M$. The eigenvalues of P^2 are real and diagonalizable. Moreover for each $x \in M$, one can observe

$$D^\lambda_x = Ker\{P^2 + \lambda^2(x)I\}_x,$$

where I denotes the identity transformation on T_xM, and $\lambda(x) \in [0,1]$ such that $-\lambda^2(x)$ is an eigenvalue of $P^2(x)$. Further, it is easy to observe that $KerF = D^1_x$ and $KerP = D^0_x$, where D^1_x is the maximal holomorphic sub space of T_xM and D^0_x is the maximal totally real subspace of T_xM, these distributions are denoted by D_T and $D^\perp$ respectively. If $-\lambda_1^2(x), \ldots, -\lambda_k^2(x)$ are the eigenvalues of P^2 at x, then T_xM can be decomposed as

$$T_xM = D^{\lambda_1}_x \oplus D^{\lambda_2}_x \oplus \ldots D^{\lambda_k}_x.$$

Every $D^{\lambda_i}_x$, $1 \leq i \leq k$ is a $P-$invariant subspace of T_xM. Moreover, if $\lambda_i \neq 0$, then $D^{\lambda_i}_x$ is even dimensional the submanifold M of a Kaehler manifold M is a generic submanifold if there exists an integer k and functions λ_i $1 \leq i \leq k$ defined on M with $\lambda_i \in (0,1)$ such that

(i) Each $-\lambda_i^2(x), 1 \leq i \leq k$, is a distinct eigenvalue of P^2 with

$$T_xM = D_x^T \oplus D_x^\perp \oplus D_x^{\lambda_1} \oplus \ldots, \oplus D_x^{\lambda_k}$$

for any $x \in M$.

(ii) The distributions of $D_x^T, D_x^\perp$ and $D_x^{\lambda_i},\ 1 \leq i \leq k$ are independent of $x \in M$.

If in addition, each λ_i is constant on M, then M is called a skew CR-submanifolds [16]. It is significant to account that CR-submanifolds are particular class of skew CR-submanifold with $k = 1, D^T = \{0\}, D^\perp = \{0\}$ and λ_1 is constant. If $D^\perp = \{0\}$, $D_T^1 \neq= \{0\}$ and $k = 1$, then M is semi-slant submanifold. Whereas if $D_T = \{0\}, D^\perp \neq \{0\}$ and $k = 1$, then M is a hemi-slant submanifold.

Definition 2.1. A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a skew CR-submanifold of order 1 if M is a skew CR-submanifold with $k = 1$ and λ_1 is constant.

We have the following result for further use

Theorem 2.1. [17] *Let M be a submanifold of an almost contact metric manifold $\bar{M}$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = -\lambda(I + \eta \otimes \xi).$$

Furthermore, if θ is slant angle, then $\lambda = \cos^2\theta$.

For any orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space T_xM, the mean curvature vector $H(x)$ and its squared norm are defined as follows

$$H(x) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),\ \|H\|^2 = \frac{1}{n^2}\sum_{i,j=1}^{n} g(h(e_i, e_i), h(e_j, e_j)), \tag{2.10}$$

where n is the dimension of M. If $h = 0$ then the submanifold is said to be totally geodesic and minimal if $H = 0$. If $h(X,Y) = g(X,Y)H$ for all $X, Y \in TM$, then M is called totally umbilical.

The scalar curvature of $\bar{M}$ is denoted by $\bar{\tau}(M)$ and is defined as

$$\tau(M) = \sum_{1 \leq p < q \leq m} \kappa_{pq}, \tag{2.11}$$

where $\kappa_{pq} = \bar{\kappa}(e_p \wedge e_q)$ and m is the dimension of the Riemannian manifold $\bar{M}$. Throughout this study, we shall use the equivalent version of the above equation, which is given by

$$2\tau(M) = \sum_{1 \leq p < q \leq m} \kappa_{pq}. \tag{2.12}$$

In a similar way, the scalar curvature $\tau(L_x)$ of a $L-$plane is given by

$$\tau(L_x) = \sum_{1 \leq p < q \leq m} \kappa_{pq}. \tag{2.13}$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_xM and if e_r belongs to the orthonormal basis $\{e_{n+1}, \ldots e_m\}$ of the normal space $T^\perp M$, then we have

$$h_{pq}^r = g(h(e_p, e_q), e_r) \tag{2.14}$$

and

$$\|h\|^2 = \sum_{p,q=1}^{n} g(h(e_p, e_q), h(e_p, e_q)). \tag{2.15}$$

Let κ_{pq} and κ_{pq} be the sectional curvatures of the plane sections spanned by e_p and e_q at x in the submanifold M^n and in the Riemannian space form $\bar{M}^m(c)$, respectively. Thus by Gauss equation, we have

$$\kappa_{pq} = \kappa_{pq} + \sum_{r=n+1}^{m} (h^r_{pp}h^r_{qq} - (h^r_{pq})^2). \tag{2.16}$$

The global tensor field for orthonormal frame of vector field $\{e_1, \ldots, e_n\}$ on M^n is defined as

$$S(X,Y) = \sum_{i=1}^{n} \{g(R(e_i, Y)Y, e_i)\}, \tag{2.17}$$

for all $X, Y \in T_xM^n$. The above tensor is called the Ricci tensor. If we fix a distinct vector e_u from $\{e_1, \ldots, e_n\}$ on M^n, which is governed by χ. Then the Ricci curvature is defined by

$$Ric(\chi) = \sum_{\substack{p=1 \\ p\neq u}}^{n} \kappa(e_p \wedge e_u). \tag{2.18}$$

For a smooth function ψ on a Riemannian manifold M with Riemannian metric g, the gradient of ψ is denoted by $\nabla\psi$ and is defined as

$$g(\nabla\psi, X) = X\psi, \tag{2.19}$$

for all $X \in TM$.

Let the dimension of M is n and $\{e_1, e_2, \ldots, e_n\}$ be a basis of TM. Then as a result of (2.19), we get

$$\|\nabla\psi\|^2 = \sum_{i=1}^{n} (e_i(\psi))^2. \tag{2.20}$$

The Laplacian of ψ is defined by

$$\Delta\psi = \sum_{i=1}^{n} \{(\nabla_{e_i} e_i)\psi - e_i e_i \psi\}. \tag{2.21}$$

For a warped product submanifold $N_1^{n_1} \times_\psi N_2^{n_2}$ isometrically immersed in a Riemannian manifold M, we observed the well known result, which is described as follows [11]

$$\sum_{p=1}^{n_1}\sum_{q=1}^{n_2} \kappa(e_p \wedge e_q) = \frac{n_2\Delta\psi}{\psi} = n_2(\Delta ln\psi - \|\nabla ln\psi\|^2). \tag{2.22}$$

3. Warped product skew CR- submanifolds

Recently, Haider and Thakur [23] demonstrated the existence of warped product skew CR-submanifolds of the type $M = N_1 \times_f N_\perp$, where N_1 is a semi-slant submanifold as defined by J. L. Cabrerizo [18] and $N_\perp$ is a totally real submanifold. Throughout this section we consider the warped product skew CR-submanifold $M = N_1 \times_f N_\perp$ in a Cosymplectic manifold M. Then it is evident that M is a proper warped product skew CR-submanifold of order 1. Moreover, the tangent space TM of M can be decompounded as follows

$$TM = D^\theta \oplus D^T \oplus D^\perp, \tag{3.1}$$

where $D^{\theta}_x = D^{\lambda_1}_x$.

Definition 3.1 The warped product $M = N_1 \times_f N_2$ isometrically immersed in a Riemannian manifold $\bar{M}$ is called N_i totally geodesic if the partial second fundamental form h_i vanishes identically. It is called N_i-minimal if the partial mean curvature vector H^i becomes zero for $i = 1, 2$.

Let $\{e_1, \ldots, e_p, e_{p+1} = \phi e_1, \ldots, e_{n_1=2p} = \phi e_p, e^1, \ldots, e^q, e^{q+1} = \sec\theta P e^1, \ldots, e^{(n_2=2q)} = \sec\theta P e^q, e^{n_2+1}, \ldots, e^{n_3}, \xi\}$ be a local orthonormal frame of vector fields such that $\{e_1, \ldots, e_p, e_{p+1} = \phi e_1, \ldots, e_{n_1=2p} = \phi e_p, \xi\}$ is an orthonormal basis of D_T, $\{e^1, \ldots, e^q, e^{q+1} = \sec\theta P e^1, \ldots, e^{(n_2=2q)} = \sec\theta P e^q\}$ is an orthonormal basis of D_θ and $\{e^{n_2+1}, \ldots, e^{n_3}\}$ is an orthonormal basis of $D^\perp$.

Throughout this paper we consider that the warped product skew CR-submanifold $M = N_1 \times_f N_\perp$ is $D-$minimal. Now we have the following lemma for further application

Lemma 3.1. *Let $M^n = N_1^{n_1+n_2} \times_f N_\perp$ be a $D-$minimal warped product skew CR-submanifold isometrically immersed in a Cosymplectic manifold $\bar{M}$, then*

$$\|H\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{m} (h^r_{n_1+1 n_1+1} + \cdots + h^r_{n_2 n_2} + \cdots + h^r_{nn})^2, \tag{3.2}$$

where $\|H\|^2$ is the squared mean curvature.

4. Ricci curvature for warped product skew CR- submanifold

In this section, we compute Ricci curvature in the expressions of squared norm of mean curvature and the warping functions as follows

Theorem 4.1. *Let $M^n = N_1^{n_1+n_2} \times_f N_\perp^{n_3}$ be a $D-$minimal warped product skew CR-submanifold isometrically immersed in a Cosymplectic space form $\bar{M}(c)$. If the holomorphic and slant distributions D and D_θ are integrable with integrable submanifolds $N_T^{n_1}$ and $N_\theta^{n_2}$ respectively. Then for each orthogonal unit vector field $\chi \in T_xM \neq \xi$, either tangent to $N_T^{n_1}$, $N_\theta^{n_2}$ or $N_\perp^{n_3}$, we have*

(1) The Ricci curvature satisfy the following inequalities

(i) If χ is tangent to $N_T^{n_1}$, then

$$\frac{1}{4} n^2 \|H\|^2 \geq Ric(\chi) + \frac{n_3 \Delta f}{f} - \frac{c}{4}(n_1 + n_1 n_2 + n_2 n_3 + n_1 n_3 - \frac{1}{2}). \tag{4.1}$$

(ii) χ is tangent to $N_\theta^{n_2}$, then

$$\frac{1}{4} n^2 \|H\|^2 \geq Ric(\chi) + \frac{n_3 \Delta f}{f} - \frac{c}{4}(n_1 + n_1 n_2 + n_2 n_3 + n_1 n_3 - 2 + \frac{3}{2}\cos^2\theta). \tag{4.2}$$

(iii) If χ is tangent to $N_\perp^{n_3}$, then

$$\frac{1}{4} n^2 \|H\|^2 \geq Ric(\chi) + \frac{n_3 \Delta f}{f} - \frac{c}{4}(n_1 + n_1 n_2 + n_2 n_3 + n_1 n_3 - 2). \tag{4.3}$$

(2) If $H(x) = 0$ for each point $x \in M^n$, then there is a unit vector field χ which satisfies the equality case of (1) if and only if M^n is mixed totally geodesic and χ lies in the relative null space N_x at x.

(3) For the equality case we have

(a) The equality case of (4.1) holds identically for all unit vector fields tangent to N_T at each $x \in M^n$ if and only if M^n is mixed totally geodesic and $D-$totally geodesic skew CR-warped product submanifold in $M^m(c)$.

(b) The equality case of (4.3) holds identically for all unit vector fields tangent to N_θ at each $x \in M^n$ if and only if M is mixed totally geodesic and either M^n is D_θ- totally geodesic skew CR-warped product submanifold or M^n is a D_θ totally umbilical in $M^m(c)$ with dim $D_\theta = 2$.

(c) The equality case of (4.2) holds identically for all unit vector fields tangent to $N_\perp^{n_2}$ at each $x \in M^n$ if and only if M is mixed totally geodesic and either M^n is $D^\perp$- totally geodesic skew CR-warped product or M^n is a $D^\perp$ totally umbilical in $M^m(c)$ with dim $D^\perp = 2$.

(d) The equality case of (1) holds identically for all unit tangent vectors to M^n at each $x \in M^n$ if and only if either M^n is totally geodesic submanifold or M^n is a mixed totally geodesic totally umbilical and $D-$ totally geodesic submanifold with dim $N_\theta = 2$ and dim $N_\perp = 2$.

Where n_1, n_2 and n_3 are the dimensions of $N_T^{n_1}$, $N_\theta^{n_2}$ and $N_\perp^{n_3}$ respectively.

Proof. Suppose that $M^n = N_1^{n_1+n_2} \times_f N_\perp^{n_3}$ be a skew CR-warped product submanifold of a Complex space form. From Gauss equation, we have

$$n^2\|H\|^2 = 2\tau(M^n) + \|h\|^2 - 2\tau(M^n). \tag{4.4}$$

Let $\{e_1, \dots, e_{n_1}, e_{n_1+1}, \dots, e_{n_2}, \dots e_n\}$ be a local orthonormal frame of vector fields on M^n such that $\{e_1, \dots, e_{n_1}\}$ is tangent to $N_T^{n_1}$, $\{e_{n_1+1}, \dots, e_{n_2}\}$ is tangent to $N_\theta^{n_2}$ and $\{e_{n_2+1}, \dots, e_n\}$ is tangent to $N_\perp^{n_3}$. So, the unit tangent vector $\chi = e_A \in \{e_1, \dots, e_n\}$ can be expanded (4.4) as follows

$$n^2\|H\|^2 = 2\tau(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}\{(h_{11}^r + \dots h_{n_2n_2}^r + \dots + h_{nn}^r - h_{AA}^r)^2 + (h_{AA}^r)^2\}$$
$$- \sum_{r=n+1}^{m}\sum_{1\le i\neq j\le n} h_{ii}^r h_{jj}^r - 2\tau(M^n). \tag{4.5}$$

The above expression can be written as follows

$$n^2\|H\|^2 = 2\tau(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}\{(h_{11}^r + \dots h_{n_2n_2}^r + \dots + h_{nn}^r)^2$$
$$+ (2h_{AA}^r - (h_{11}^r + \dots + h_{nn}^r))^2\} + 2\sum_{r=n+1}^{m}\sum_{1\le i<j\le n}(h_{ij}^r)^2$$
$$- 2\sum_{r=n+1}^{m}\sum_{\substack{1\le i<j\le n\\ i,j\neq A}} h_{ii}^r h_{jj}^r - 2\tau(M^n).$$

In view of the assumption that skew CR-warped product submanifold $M = N_1 \times_f N_\perp$ is $D-$minimal submanifold. Then the preceding expression takes the form,

$$n^2\|H\|^2 = 2\tau(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}\{(h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn})^2$$
$$+ \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{AA} - (h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn}))^2 \tag{4.6}$$
$$+ 2\sum_{r=n+1}^{m}\sum_{1\le i<j\le n}(h^r_{ij})^2 - 2\sum_{r=n+1}^{m}\sum_{\substack{1\le i<j\le n\\ i,j\neq A}} h^r_{ii}h^r_{jj} - 2\tau(M^n).$$

Considering unit tangent vector $\chi = e_A$, we have three choices χ is either tangent to the base manifolds $N_T^{n_1}$, $N_\theta^{n_2}$ or to the fiber $N_\perp^{n_3}$.

Case 1: If χ is tangent to $N_T^{n_1}$, then we need to choose a unit vector field from $\{e_1, \dots, e_{n_1}\}$. Let $\chi = e_1$, then by (2.17) and the assumption that the submanifolds is $D-$minimal, we have

$$n^2\|H\|^2 \ge Ric(\chi) + \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn})^2$$
$$+ \frac{n_3\Delta f}{f} + \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{11} - (h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn}))^2$$
$$+ \sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le n_1}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2)$$
$$+ \sum_{r=n+1}^{m}\sum_{n_1+1\le p<q\le n_2}(h^r_{pp}h^r_{qq} - (h^r_{pq})^2) \tag{4.7}$$
$$+ \sum_{r=n+1}^{m}\sum_{n_2+1\le s<t\le n}(h^r_{ss}h^r_{tt} - (h^r_{st})^2)$$
$$+ \sum_{r=n+1}^{m}\sum_{1\le i<j\le n}(h^r_{ij})^2 - \sum_{r=n+1}^{m}\sum_{2\le i<j\le n}(h^r_{ii}h^r_{jj})$$
$$- 2\bar{\tau}(M) + \sum_{2\le i<j\le n}\bar{\kappa}(e_i, e_j) + \bar{\tau}(N_T^{n_1}) + \bar{\tau}(N_\theta^{n_2}) + \bar{\tau}(N_\perp^{n_3}).$$

Putting $X = W = e_i, Y, Z = e_j$ in the formula (2.2), we have

$$2\bar{\tau}(\check{M}) = \frac{c}{4}(n(n-1)) - \frac{c}{4}(2(n-1) - 3(n_1-1) - 3n_2\cos^2\theta) \tag{4.8}$$
$$\sum_{2\le i<j\le n}\kappa(e_i, e_j) = \frac{c}{8}((n-1)(n-2)) - \frac{c}{8}(2(n-2) - 3(n_1-2) - 3n_2\cos^2\theta)$$
$$\bar{\tau}(N_T^{n_1}) = \frac{c}{8}(n_1(n_1-1) + 3n_1) - \frac{c}{8}(2(n_1-1) - 3(n_1-1))$$
$$\bar{\tau}(N_\theta^{n_2}) = \frac{c}{8}(n_2(n_2-1)) - \frac{\hat{c}}{8}(-3n_2\cos^2\theta).$$
$$\bar{\tau}(N_\perp^{n_3}) = \frac{c-3}{8}(n_3(n_3-1))$$

Using these values in (4.7), we get

$$\begin{aligned}n^2\|H\|^2 \geq& Ric(\chi) + \frac{1}{2}n^2\|H\|^2 + \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{11} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2\\&+ \frac{n_3\Delta f}{f} + \sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_2}(h^r_{ij})^2\\&+ \sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{k=n_2+1}^{n}(h^r_{ik})^2 + \sum_{r=n+1}^{m}\sum_{\beta=2}^{n_1}h^r_{11}h^r_{\beta\beta}\\&- \sum_{r=n+1}^{m}\sum_{i=2}^{n_1}\sum_{j=n_1+1}^{n_2}h^r_{ii}h^r_{jj} - \sum_{r=n+1}^{m}\sum_{i=2}^{n_1}\sum_{k=n_2+1}^{n}h^r_{ii}h^r_{kk}\\&- \frac{c}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - \frac{5}{2}).\end{aligned} \tag{4.9}$$

In view of the assumption that the submanifold is $D-$minimal, then

$$\sum_{r=n+1}^{m}\sum_{\beta=2}^{n_1}h^r_{11}h^r_{\beta\beta} = \sum_{r=n+1}^{m}(h^r_{11})^2$$

$$-\sum_{r=n+1}^{m}\sum_{i=2}^{n_1}[\sum_{j=n_1+1}^{n_2}h^r_{ii}h^r_{jj} + \sum_{k=n_2+1}^{n}h^r_{ii}h^r_{kk}] = \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}h^r_{11}h^r_{jj}.$$

Utilizing in (4.9), we have

$$\begin{aligned}n^2\|H\|^2 \geq& Ric(\chi) + \frac{1}{2}n^2\|H\|^2 + \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{11} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2\\&+ \frac{n_3\Delta f}{f} + \sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_2}(h^r_{ij})^2\\&+ \sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{k=n_2+1}^{n}(h^r_{ik})^2 - \sum_{r=n+1}^{m}(h^r_{11})^2 + \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}h^r_{ii}h^r_{jj}\\&- \frac{c}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - \frac{5}{2}).\end{aligned} \tag{4.10}$$

The third term on the right hand side can be written as

$$\begin{aligned}&\frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{11} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{n_2n_2} + \cdots + h^r_{nn}))^2\\&\qquad = 2\sum_{r=n+1}^{m}(h^r_{11})^2 + \frac{1}{2}n^2\|H\|^2 - 2\sum_{r=n+1}^{m}[\sum_{j=n_1+1}^{n_2}h^r_{11}h^r_{jj}\\&\qquad + \sum_{k=n_2+1}^{n}h^r_{11}h^r_{kk}].\end{aligned} \tag{4.11}$$

Combining above two expressions, we have

$$\begin{aligned}\frac{1}{2}n^2\|H\|^2 \geq & Ric(\chi) + \sum_{r=n+1}^{m}(h_{11}^r)^2 - \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n} h_{11}^r h_{jj}^r \\ & + \frac{1}{2}\sum_{r=n+1}^{m}(h_{n_1+1n_1+1}^r + \cdots + h_{n_2n_2}^r + \cdots + h_{nn}^r)^2 \\ & + \sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}(h_{ij}^r)^2 + \frac{n_3\Delta f}{f} \\ & - \frac{c}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - \frac{5}{2}).\end{aligned} \tag{4.12}$$

Or equivalently

$$\begin{aligned}\frac{1}{4}n^2\|H\|^2 \geq & Ric(\chi) + \frac{1}{4}\sum_{r=n+1}^{m}(2h_{11}^r - (h_{n_1+1n_1+1}^r + \cdots + h_{n_2n_2}^r + \cdots + h_{nn}^r))^2 \\ & + \sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}(h_{ij}^r)^2 + \frac{n_3\Delta f}{f} \\ & - \frac{c}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - \frac{5}{2}).\end{aligned} \tag{4.13}$$

Which gives the inequality (i) of (1).
Case 2. If χ is tangent to $N_\theta^{n_2}$, we choose the unit vector from $\{e_{n_1+1}, \ldots, e_{n_2}\}$. Suppose $\chi = e_{n_2}$, then from (4.6), we deduce

$$\begin{aligned}n^2\|H\|^2 \geq & Ric(\chi) + \frac{1}{2}\sum_{r=n+1}^{m}(h_{n_1+1n_1+1}^r + \ldots h_{n_2n_2}^r + \cdots + h_{nn}^r)^2 \\ & + \frac{n_3\Delta f}{f} + \frac{1}{2}\sum_{r=n+1}^{m}((h_{n_1+1n_1+1}^r + \ldots h_{n_2n_2}^r + \cdots + h_{nn}^r) - 2h_{n_2n_2}^r)^2 \\ & + \sum_{r=n+1}^{m}\sum_{1\leq\alpha<\beta\leq n_1}(h_{\alpha\alpha}^r h_{\beta\beta}^r - (h_{\alpha\beta}^r)^2) + \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n_2}(h_{ss}^r h_{tt}^r - (h_{st}^r)^2) \\ & + \sum_{r=n+1}^{m}\sum_{n_2+1\leq p<q\leq n}(h_{pp}^r h_{qq}^r - (h_{pq}^r)^2) + \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}(h_{ij}^r)^2 \\ & - \sum_{r=n+1}^{m}\sum_{\substack{1\leq i<j\leq n\\ i,j\neq n_2}}(h_{ii}^r h_{jj}^r) - 2\bar{\tau}(M) + \sum_{\substack{1\leq i<j\leq n\\ i,j\neq n_2}}\bar{\kappa}(e_i, e_j) \\ & + \bar{\tau}(N_T^{n_1}) + \bar{\tau}(N_\theta^{n_2} + \bar{\tau}(N_\perp^{n_3})).\end{aligned} \tag{4.14}$$

From (2.2) by putting $X = W = e_i, W, Z = e_j$, one can compute

$$\sum_{\substack{1\leq i<j\leq n\\ i,j\neq n_2}}\kappa(e_i, e_j) = \frac{c}{8}((n-1)(n-2)) - \frac{c}{8}(2(n-2) - 3(n_1-1) - 3n_2\cos^2\theta)$$

$$\bar{\tau}(N_T^{n_1}) = \frac{c}{8}(n_1(n_1-1)) - \frac{c}{8}(2(n_1-1) - 3(n_1-1))$$

$$\bar{\tau}(N_\theta^{n_2}) = \frac{c}{8}(n_2(n_2-1)) - \frac{c}{8}(-3n_2\cos^2\theta).$$
$$\bar{\tau}(N_\perp^{n_3}) = \frac{c}{8}(n_3(n_3-1))$$

Using these values together with (4.8) in (4.14) and applying similar techniques as in Case 1, we obtain

$$\begin{aligned} n^2\|H\|^2 \geq & Ric(\chi) + \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1}+\dots h^r_{n_2n_2}+\dots+h^r_{nn}) - 2h^r_{n_2n_2}))^2 \\ & + \frac{1}{2}n^2\|H\|^2 + \frac{n_3\Delta f}{f} + \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}(h^r_{ij})^2 \\ & + \sum_{r=n+1}^{m}[\sum_{t=n_1+1}^{n_2-1}h^r_{n_2n_2}h^r_{tt} + \sum_{l=n_2+1}^{n}h^r_{n_2n_2}h^r_{ll}] \\ & \sum_{r=1}^{m}\sum_{i=1}^{n_1}[\sum_{j=n_1+1}^{n_2-1}h^r_{ii}h^r_{jj} + \sum_{k=n_2+1}^{n}h^r_{ii}h^r_{kk}] \\ & - \frac{c}{4}(n+n_1n_2+n_2n_3+n_1n_3-4+\frac{3}{2}\cos^2\theta). \end{aligned} \tag{4.15}$$

By the assumption that the submanifold M^n is $D-$minimal, one can conclude

$$\sum_{r=1}^{m}\sum_{i=1}^{n_1}[\sum_{j=n_1+1}^{n_2-1}h^r_{ii}h^r_{jj} + \sum_{k=n_2+1}^{n}h^r_{ii}h^r_{kk}] = 0.$$

The second and seventh terms on right hand side of (4.15) can be solved as follows

$$\begin{aligned} & \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1}+\dots+h^r_{nn}) - 2h^r_{n_2n_2}))^2 + \sum_{r=n+1}^{m}[\sum_{t=n_1+1}^{n_2-1}h^r_{n_2n_2}h^r_{tt} + \sum_{l=n_2+1}^{n}h^r_{n_2n_2}h^r_{ll}] \\ & = \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1}+\dots+h^r_{nn})^2 + 2\sum_{r=n+1}^{m}(h^r_{n_2n_2})^2 \\ & - 2\sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}h^r_{n_2n_2}h^r_{jj} + \sum_{r=n+1}^{m}\sum_{t=n_1+1}^{n}h^r_{n_2n_2}h^r_{tt} - \sum_{r=n+1}^{m}(h^r_{n_2n_2})^2 \\ & = \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1}+\dots+h^r_{nn})^2 + \sum_{r=n+1}^{m}(h^r_{n_2n_2})^2 \\ & - \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}h^r_{nn}h^r_{jj}. \end{aligned} \tag{4.16}$$

Utilizing these two values in (4.15), we arrive

$$\begin{aligned}\frac{1}{2}n^2\|H\|^2 \geq & Ric(\chi) + \sum_{r=n+1}^{m}(h^r_{n_2n_2})^2 - \sum_{r=n+1}^{m}\sum_{i=n_1+1}^{n} h^r_{nn}h^r_{jj} \\ & + \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \cdots + h^r_n n)^2 + \frac{1}{2}n^2\|H\|^2 + \frac{n_3\Delta f}{f} \\ & + \sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}(h^r_{ij})^2 - \frac{c}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 4 + \frac{3}{2}\cos^2\theta).\end{aligned} \tag{4.17}$$

By using similar steps as in Case 1, the above inequality can be written as

$$\begin{aligned}\frac{1}{4}n^2\|H\|^2 \geq & Ric(\chi) + \frac{1}{4}\sum_{r=n+1}^{m}(2h^r_{n_2n_2} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2 \\ & + \frac{n_3\Delta f}{f} - \frac{c}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 4 + \frac{3}{2}\cos^2\theta).\end{aligned} \tag{4.18}$$

The last inequality leads to inequality (ii) of (1).

Case 3. If χ is tangent to $N^{n_3}_{\perp}$, then we choose the unit vector field from $\{e_{n_2+1}, \ldots, e_n\}$. Suppose the vector χ is e_n. Then from (4.6)

$$\begin{aligned}n^2\|H\|^2 \geq & Ric(\chi) + \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \ldots h^r_{n_2n_2} + \cdots + h^r_{nn})^2 \\ & + \frac{n_3\Delta f}{f} + \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \ldots h^r_{n_2n_2} + \cdots + h^r_{nn}) - 2h^r_{nn})^2 \\ & + \sum_{r=n+1}^{m}\sum_{1\leq\alpha<\beta\leq n_1}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2) + \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n_2}(h^r_{ss}h^r_{tt} - (h^r_{st})^2) \\ & + \sum_{r=n+1}^{m}\sum_{n_2+1\leq p<q\leq n}(h^r_{pp}h^r_{qq} - (h^r_{pq})^2) + \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}(h^r_{ij})^2 \\ & - \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n-1}h^r_{ii}h^r_{jj} - 2\bar{\tau}(M) + \sum_{1\leq i<j\leq n-1}\bar{\kappa}(e_i, e_j) \\ & + \bar{\tau}(N^{n_1}_T) + \bar{\tau}(N^{n_2}_\theta) + \bar{\tau}(N^{n_3}_\perp).\end{aligned} \tag{4.19}$$

From (2.2), one can compute

$$\sum_{1\leq i<j\leq n-1}\kappa(e_i, e_j) = \frac{c}{8}((n-1)(n-2)) - \frac{c}{8}(2(n-2) - 3(n_1-1) - 3(n_2-1)\cos^2\theta)$$

$$\bar{\tau}(N^{n_1}_T) = \frac{c}{8}(n_1(n_1-1)) - \frac{c}{8}(2(n_1-1) - 3(n_1-1))$$

$$\bar{\tau}(N^{n_2}_\theta) = \frac{c}{8}(n_2(n_2-1)) - \frac{c}{8}(-3n_2\cos^2\theta).$$

$$\bar{\tau}(N^{n_3}_\perp) = \frac{c}{8}(n_3(n_3-1))$$

By usage of these values together with (4.8) in (4.19) and analogous to Case 1 and Case 2, we obtain

$$\begin{aligned} n^2\|H\|^2 \geq & Ric(\chi) + \frac{1}{2}n^2\|H\|^2 + \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn}) - 2h^r_{nn})^2 \\ & + \frac{n_3\Delta f}{f} + \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}(h^r_{ij})^2 \\ & + \sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n-1} h^r_{nn}h^r_{qq} - \sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n-1} h^r_{ii}h^r_{jj} \\ & - \frac{c}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 + 1 - 4). \end{aligned} \tag{4.20}$$

Again using the assumption that M^n is $D - minimal$ and it is easy to verify

$$\sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n-1} h^r_{ii}h^r_{jj} = 0. \tag{4.21}$$

Using in (4.20), we obtain

$$\begin{aligned} n^2\|H\|^2 \geq & Ric(\chi) + \frac{1}{2}n^2\|H\|^2 + \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn}) - 2h^r_{nn})^2 \\ & + \frac{n_3\Delta f}{f} + \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}(h^r_{ij})^2 + \sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n-1} h^r_{nn}h^r_{qq} \\ & - \frac{c}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 4). \end{aligned} \tag{4.22}$$

The third and sixth terms on the right hand side of (4.22) in a similar way as in Case 1 and Case 2 can be simplified as

$$\begin{aligned} & \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn}) - 2h^r_{nn})^2 + \sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n-1} h^r_{nn}h^r_{qq} \\ & = \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn})^2 + \sum_{r=n+1}^{m}(h^r_{nn})^2 \\ & - \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n} h^r_{nn}h^r_{jj}. \end{aligned} \tag{4.23}$$

By combining (4.22) and (4.23) and using similar techniques as used in Case 1 and Case 2, we can derive

$$\begin{aligned} \frac{1}{4}n^2\|H\|^2 \geq & Ric(\chi) + \frac{1}{4}\sum_{r=n+1}^{m}(2h^r_{nn} - (h^r_{n_1+1n_1+1} + \dots + h^r_{nn}))^2 \\ & + \frac{n_3\Delta f}{f} - \frac{c}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 4). \end{aligned} \tag{4.24}$$

The last inequality leads to inequality (iii) in (1).

Next, we explore the equality cases of (1). First, we redefine the notion of the relative null space N_x of the submanifold M^n in the Complex space form $\bar{M}^m(c)$ at any point $x \in M^n$, the relative null space was defined by B. Y. Chen [9], as follows

$$N_x = \{X \in T_xM^n : h(X,Y) = 0, \forall Y \in T_xM^n\}.$$

For $A \in \{1,\ldots,n\}$ a unit vector field e_A tangent to M^n at x satisfies the equality sign of (4.1) identically if and only if

$$(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0 \ (ii)\ \sum_{b=1}^{n}\sum_{\substack{A=1\\ b\neq A}}^{n} h^r_{bA} = 0 \ (iii)\ 2h^r_{AA} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{4.25}$$

such that $r \in \{n+1,\ldots m\}$ the condition (i) implies that M^n is mixed totally geodesic skew CR-warped product submanifold. Combining statements (ii) and (iii) with the fact that M^n is skew CR-warped product submanifold, we get that the unit vector field $\chi = e_A$ belongs to the relative null space N_x. The converse is trivial, this proves statement (2).

For a skew CR-warped product submanifold, the equality sign of (4.1) holds identically for all unit tangent vector belong to $N_T^{n_1}$ at x if and only if

$$(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0 \ (ii)\sum_{b=1}^{n}\sum_{\substack{A=1\\ b\neq A}}^{n_1} h^r_{bA} = 0 \ (iii)\ 2h^r_{pp} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{4.26}$$

where $p \in \{1,\ldots,n_1\}$ and $r \in \{n+1,\ldots,m\}$. Since M^n is $D-$minimal skew CR-warped product submanifold, the third condition implies that $h^r_{pp} = 0,\ p \in \{1,\ldots,n_1\}$. Using this in the condition (ii), we conclude that M^n is $D-$totally geodesic skew CR-warped product submanifold in $\bar{M}^m(c)$ and mixed totally geodesicness follows from the condition (i). Which proves (a) in the statement (3).

For a skew CR-warped product submanifold, the equality sign of (4.2) holds identically for all unit tangent vector fields tangent to $N_\theta^{n_2}$ at x if and only if

$$(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0 \ (ii)\sum_{b=1}^{n}\sum_{\substack{A=n_1+1\\ b\neq A}}^{n_2} h^r_{bA} = 0 \ (iii)\ 2h^r_{KK} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{4.27}$$

such that $K \in \{n_1+1,\ldots,n_2\}$ and $r \in \{n+1,\ldots,m\}$. From the condition (iii) two cases emerge, that is

$$h^r_{KK} = 0,\ \forall K \in \{n_1+1,\ldots,n_2\} \text{ and } r \in \{n+1,\ldots,m\} \text{ or } dim\ N_\theta^{n_2} = 2. \tag{4.28}$$

If the first case of (4.27) satisfies, then by virtue of condition (ii), it is easy to conclude that M^n is a $D_\theta-$ totally geodesic skew CR-warped product submanifold in $\bar{M}^m(c)$. This is the first case of part (b) of statement (3).

For a skew CR-warped product submanifold, the equality sign of (4.3) holds identically for all unit tangent vector fields tangent to $N_\perp^{n_3}$ at x if and only if

$$(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0 \ (ii)\sum_{b=1}^{n}\sum_{\substack{A=n_2+1\\ b\neq A}}^{n_3} h^r_{bA} = 0 \ (iii)\ 2h^r_{LL} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{4.29}$$

such that $L \in \{n_2+1,\ldots,n\}$ and $r \in \{n+1,\ldots,m\}$. From the condition (iii) two cases arise, that is

$$h^r_{LL} = 0,\ \forall L \in \{n_2+1,\ldots,n\} \text{ and } r \in \{n+1,\ldots,m\} \text{ or } dim\ N_\perp^{n_3} = 2. \tag{4.30}$$

If the first case of (4.29) satisfies, then by virtue of condition (ii), it is easy to conclude that M^n is a $D_\perp-$ totally geodesic skew CR-warped product submanifold in $\bar{M}^m(c)$. This is the first case of part (c) of statement (3).

For the other case, assume that M^n is not $D_\perp-$totally geodesic skew CR-warped product submanifold and dim $N_\perp^{n_3} = 2$. Then condition (ii) of (4.29) implies that M^n is $D_\perp-$ totally umbilical skew CR-warped product submanifold in $\bar{M}(c)$, which is second case of this part. This verifies part (c) of (3).

To prove (d) using parts (a), (b) and (c) of (3), we combine (4.26),(4.27) and (4.29). For the first case of this part, assume that $dimN_\theta^{n_2} \neq 2$ and $dimN_\perp^{n_3} \neq 2$. Since from parts (a), (b) and (c) of statement (3) we conclude that M^n is $D-$totally geodesic, $D_\theta-$ totally geodesic and $D_\perp-$ totally geodesic submanifolds in $\bar{M}^m(c)$. Hence M^n is a totally geodesic submanifold in $\bar{M}^m(c)$.

For another case, suppose that first case does not satisfy. Then parts (a), (b) and (c) provide that M^n is mixed totally geodesic and $D-$ totally geodesic submanifold of $\bar{M}^m(c)$ with $dimN_\theta = 2$ and $dimN_\perp = 2$. From the conditions (b) and (c) it follows that M^n is $D_\theta-$ and $D_\perp-$totally umbilical skew CR-warped product submanifolds and from (a) it is $D-$totally geodesic, which is part (d). This proves the theorem. □

In view of (2.22) we have the another version of the theorem 2 as follows

Theorem 4.2. *Let $M^n = N_1^{n_1+n_2} \times_f N_\perp^{n_3}$ be a $D-$minimal warped product skew CR-submanifold isometrically immersed in a Cosymplectic space form $\bar{M}(c)$. If the holomorphic and slant distributions D and D_θ are integrable with integral submanifolds $N_T^{n_1}$ and $N_\theta^{n_2}$ respectively. Then for each orthogonal unit vector field $\chi \in T_xM$, either tangent to $N_T^{n_1}$, $N_\theta^{n_2}$ or $N_\perp^{n_3}$, we have*

(1) The Ricci curvature satisfy the following inequalities

(i) If χ is tangent to $N_T^{n_1}$, then

$$\frac{1}{4}n^2\|H\|^2 \geq Ric(\chi) + n_3(\Delta lnf - \|\nabla lnf\|^2) - \frac{c}{4}(n_1 + n_1n_2 + n_2n_3 + n_1n_3 - \frac{1}{2}). \tag{4.31}$$

(ii) χ is tangent to $N_\theta^{n_2}$, then

$$\frac{1}{4}n^2\|H\|^2 \geq Ric(\chi) + n_3(\Delta lnf - \|\nabla lnf\|^2) - \frac{c}{4}(n_1 + n_1n_2 + n_2n_3 + n_1n_3 - 2 + \frac{3}{2}\cos^2\theta). \tag{4.32}$$

(iii) If χ is tangent to $N_\perp^{n_2}$, then

$$\frac{1}{4}n^2\|H\|^2 \geq Ric(\chi) + n_3(\Delta lnf - \|\nabla lnf\|^2) - \frac{c}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 2). \tag{4.33}$$

(2) If $H(x) = 0$ for each point $x \in M^n$, then there is a unit vector field χ which satisfies the equality case of (1) if and only if M^n is mixed totally geodesic and χ lies in the relative null space N_x at x.

(3) For the equality case we have

(a) The equality case of (4.1) holds identically for all unit vector fields tangent to N_T at each $x \in M^n$ if and only if M^n is mixed totally geodesic and $D-$totally geodesic skew CR-warped product submanifold in $\bar{M}^m(c)$.

(b) The equality case of (4.3) holds identically for all unit vector fields tangent to N_θ at each $x \in M^n$ if and only if M is mixed totally geodesic and either

M^n is D_θ- totally geodesic skew CR-warped product submanifold or M^n is a D_θ totally umbilical in $\bar{M}^m(c)$ with dim $D_\theta = 2$.

(c) The equality case of (4.2) holds identically for all unit vector fields tangent to $N_\perp^{n_2}$ at each $x \in M^n$ if and only if M is mixed totally geodesic and either M^n is $D^\perp$- totally geodesic skew CR-warped product or M^n is a $D^\perp$ totally umbilical in $\bar{M}^m(c)$ with dim $D^\perp = 2$.

(d) The equality case of (1) holds identically for all unit tangent vectors to M^n at each $x \in M^n$ if and only if either M^n is totally geodesic submanifold or M^n is a mixed totally geodesic totally umbilical and $D-$ totally geodesic submanifold with dim $N_\theta = 2$ and dim $N_\perp = 2$.

Where n_1, n_2 and n_3 are the dimensions of $N_T^{n_1}$, $N_\theta^{n_2}$ and $N_\perp^{n_3}$ respectively.

COMPETING INTERESTS

All the authors declares that they have no competing interests.

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AUTHORS CONTRIBUTIONS

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