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EXISTENCE OF SOLUTIONS FOR HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

ABDELOUAHEB ARDJOUNI

ABSTRACT. In this work, we investigate the existence of solutions for Hadamard fractional differential equations with integral boundary conditions in a Banach space. We will make use the measure of noncompactness and the Mönch fixed point theorem to prove the main results. An example is given to illustrate our results.

1. INTRODUCTION

Fractional differential equations are attracting attention of researchers because of the fact that fractional order derivatives are the better to describe many different natural phenomena compared with classical integer order derivatives. Therefore, initial and boundary value problems including fractional differential equations are more appropriate for many mathematical models of various fields of science and engineering. In particular, problems concerning qualitative analysis of the positivity and stability of such solutions for fractional differential equations have received the attention of many authors, see [1]-[11], [13]-[33] and the references therein.

In [21], Lachouri, Ardjouni and Djoudi discussed the existence of solutions for the following fractional differential equation in a Banach space

$$\begin{cases} D^{\alpha}x(t) - f(t,x(t)) = D^{\alpha-1}g(t,x(t)),\ t\in(0,1), \\ x(0)=0,\ x(1)=\int_0^1 g(s,x(s))\,ds, \end{cases}$$

where D^{α} is the standard Riemann-Liouville fractional derivative of order $1<\alpha\leq 2$, and $f,g:[0,1]\times E\to E$ are given continuous functions. By employing the measure of noncompactness and the Mönch fixed point theorem, the authors obtained existence results.

In this paper, we extend the results in [21] by proving the existence of solutions for the following Hadamard fractional differential equation in a Banach space

$$\begin{cases} \mathfrak{D}^{\alpha}x(t) - f(t,x(t)) = \mathfrak{D}^{\alpha-1}g(t,x(t)),\ t\in(1,e), \\ x(1)=0,\ x(e)=\int_1^e g(s,x(s))\,\frac{ds}{s}, \end{cases} \tag{1.1}$$

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where $\mathfrak{D}^{\alpha}$ is the Hadamard fractional derivative of order $1 < \alpha \leq 2$, $f, g : [1, e] \times E \to E$ are given continuous functions satisfying some assumptions that will be specified later, and E is a Banach space with the norm $\|.\|$. To prove the existence of solutions, we transform (1.1) into an equivalent integral equation and then by the measure of noncompactness and use the Mönch fixed point theorem.

2. Preliminaries

Let $J = [1, e]$. By $C(J, E)$ we denote the Banach space of all continuous functions from J into E with the norm

$$\|x\|_{\infty} = \sup \{\|x(t)\| : t \in J\}.$$

Let $L^1(J, E)$ be the Banach space of measurable functions $x : J \to E$ that are Lebesgue integrable with norm

$$\|x\|_{L^1} = \int_J \|x(t)\| \, dt.$$

And $AC(J, E)$ be the space of absolutely continuous valued functions on J, and set

$$AC^n(J, E) = \left\{x : J \to E : x, x', x'', \quad , x^{n-1} \in C(J, E) \text{ and } x^{n-1} \in AC(J, E)\right\}.$$

Moreover, for a given set V of function $v : J \to E$, let us denote by

$$V(t) = \{v(t) : v \in V\}, \ t \in J,$$

and

$$V(J) = \{v(t) : v \in V, \ t \in J\}.$$

Definition 2.1 ([20]). The Hadamard fractional integral of order $\alpha > 0$ of a continuous function $x : J \to E$ is given by

$$\mathfrak{I}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} x(s) \frac{ds}{s},$$

provided the right side is pointwise defined on J.

Definition 2.2 ([20]). The Hadamard fractional derivative of order α of a continuous function $x : J \to E$ is defined by

$$\mathfrak{D}^{\alpha} x(t) = \frac{1}{\Gamma(n - \alpha)} \left(t \frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n - \alpha - 1} x(s) \frac{ds}{s},$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of real number α.

Lemma 2.1 ([20]). *The solution of the linear Hadamard fractional differential equation*

$$\mathfrak{D}^{\alpha} x(t) = 0,$$

is given by

$$x(t) = c_1 (\log t)^{\alpha - 1} + c_2 (\log t)^{\alpha - 2} + c_3 (\log t)^{\alpha - 3} + \ldots + c_n (\log t)^{\alpha - n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ *and* $n = [\alpha] + 1$.

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.3 ([5, 12]). Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_E \to [0, \infty)$ defined by

$$\mu(B) = \inf \{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and } diam(B_i) \leq \epsilon\}, \text{ here } B \in \Omega_E.$$

The measure of noncompactness satisfies some important properties
(a) $\mu(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact),
(b) $\mu(B) = \mu(\overline{B})$,
(c) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$,
(d) $\mu(A + B) \leq \mu(A) + \mu(B)$,
(e) $\mu(cB) = |c|\, \mu(B)\,,\, c \in \mathbb{R}$,
(f) $\mu(convB) = \mu(B)$.
Here $\overline{B}$ and $convB$ denote the closure and the convex hull of the bounded set B, respectively. The details of μ and its properties can be found in [5, 12].

Definition 2.4. A map $f : J \times E \to E$ is said to be Caratheodory if
(i) $t \to f(t, x)$ is measurable for each $x \in E$.
(ii) $x \to f(t, x)$ is continuous for almost all $t \in J$.

To prove the existence of solutions of (1.1), we need the following results.

Theorem 2.2 ([4]). *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \text{ or } V = N(V) \cup \{0\} \Rightarrow \mu(V) = 0,$$

holds for every V of D, then N has a fixed point.

Lemma 2.3 ([27]). *Let D be a bounded, closed and convex subset of the Banach space $C(J, E)$. Let G be a continuous function on $J \times J$ and f a function from $J \times E \to E$, which satisfies the Carathéodory conditions, and assume there exists $p \in L^1(J, \mathbb{R}^+)$ such that, for each $t \in J$ and each bounded set $B \subset E$, we have*

$$\lim_{h \to 0^+} \mu(f(J_{t,h} \times B)) \leq p(t)\, \mu(B)\,, \text{ here } J_{t,h} = [t - h, t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\mu\left(\left\{\int_J G(t, s)\, f(s, y(s))\, \frac{ds}{s} : y \in V\right\}\right) \leq \int_J \|G(t, s)\|\, p(s)\, \mu(V(s))\, \frac{ds}{s}.$$

3. Existence results

Let us start by defining what we mean by a solution of the problem (1.1).

Definition 3.1. A function $x \in AC^2(J, E)$ is said to be a solution of problem (1.1) if x satisfies the equation $\mathfrak{D}^{\alpha} x(t) - f(t, x(t)) = \mathfrak{D}^{\alpha-1} g(t, x(t))$ on J and the conditions $x(1) = 0$, $x(e) = \int_1^e g(s, x(s))\, ds$.

For the existence of solutions for the problem (1.1), we need the following auxiliary lemma.

Lemma 3.1. *The function x solves the problem (1.1) if and only if it is a solution of the integral equation*

$$x(t) = \int_1^e G(t, s)\, f(s, x(s))\, \frac{ds}{s} + \int_1^t g(s, x(s))\, \frac{ds}{s},\ t \in J,$$

where G is the Green function given by

$$G(t, s) = \begin{cases} \frac{\left[(\log t)\left(\log \frac{e}{s}\right)\right]^{\alpha-1} - \left(\log \frac{t}{s}\right)^{\alpha-1}}{\Gamma(\alpha)},\ 1 \leq s \leq t \leq e, \\ \frac{\left[(\log t)\left(\log \frac{e}{s}\right)\right]^{\alpha-1}}{\Gamma(\alpha)},\ 1 \leq t \leq s \leq e. \end{cases}$$

Proof. From Lemma 2.1, applying the Hadamard fractional integral $\mathfrak{I}^{\alpha}$ on both sides of (1.1), we have

$$\begin{aligned}
&x(t) - c_1(\log t)^{\alpha-1} - c_2(\log t)^{\alpha-2} + \mathfrak{I}^{\alpha} f(t, x(t)) \\
&= \mathfrak{I}\left(\mathfrak{I}^{\alpha-1}\mathfrak{D}^{\alpha-1} g(t, x(t))\right) \\
&= \mathfrak{I}\left(g(t, x(t)) - c_3(\log t)^{\alpha-2}\right).
\end{aligned}$$

That is,

$$\begin{aligned}
x(t) &= c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2} - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s, x(s))\frac{ds}{s} \\
&\quad + \int_1^t g(s, x(s))\frac{ds}{s} - \frac{c_3}{\alpha-1}(\log t)^{\alpha-1}.
\end{aligned}$$

By the boundary conditions $x(1) = 0$, $x(e) = \int_1^e g(s, x(s))\,ds$, one has $c_2 = 0$ and

$$c_1 = \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1} f(s, x(s))\frac{ds}{s} + \frac{c_3}{\alpha-1}.$$

Therefore

$$\begin{aligned}
x(t) &= \frac{1}{\Gamma(\alpha)}\int_1^e (\log t)^{\alpha-1}\left(\log\frac{e}{s}\right)^{\alpha-1} f(s, x(s))\frac{ds}{s} \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s, x(s))\frac{ds}{s} + \int_1^t g(s, x(s))\frac{ds}{s} \\
&= \int_1^e G(t, s) f(s, x(s))\frac{ds}{s} + \int_1^t g(s, x(s))\frac{ds}{s}.
\end{aligned}$$

This process is reversible. The proof is complete. □

In the following, we prove existence results for the boundary value problem (1.1) by using the Mönch fixed point theorem.

The following assumptions will be used in our main results

(H1) The functions $f, g : J \times E \to E$ satisfy the Caratheodory conditions.

(H2) There exist $p_f, p_g \in L^1(J, \mathbb{R}^+) \cap C(J, \mathbb{R}^+)$ such that

$$\begin{aligned}
\|f(t, x)\| &\leq p_f(t)\|x\|, \text{ for } t \in J \text{ and each } x \in E, \\
\|g(t, x)\| &\leq p_g(t)\|x\|, \text{ for } t \in J \text{ and each } x \in E.
\end{aligned}$$

(H3) For each $t \in J$ and each bounded set $B \subset E$, we have

$$\begin{aligned}
\lim_{h\to 0^+}\mu(f(J_{t,h} \times B)) &\leq p_f(t)\mu(B), \text{ here } J_{t,h} = [t-h, t] \cap J, \\
\lim_{h\to 0^+}\mu(g(J_{t,h} \times B)) &\leq p_g(t)\mu(B), \text{ here } J_{t,h} = [t-h, t] \cap J.
\end{aligned}$$

Theorem 3.2. *Assume that the assumptions (H1)-(H3) hold. If*

$$\frac{2}{\Gamma(\alpha+1)}\|p_f\|_{\infty} + \|p_g\|_{\infty} < 1, \tag{3.1}$$

then the boundary value problem (1.1) has at least one solution.

Proof. We transform the problem (1.1) into a fixed point problem by defining an operator $N : C(J, E) \to C(J, E)$ as

$$(Nx)(t) = \frac{1}{\Gamma(\alpha)} \int_1^e (\log t)^{\alpha-1} \left(\log \frac{e}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} + \int_1^t g(s, x(s)) \frac{ds}{s},$$

Clearly, the fixed points of operator N are solutions of the problem (1.1). Let $R > 0$ and consider the set

$$D_R = \{x \in C(J, E) : \|x\|_\infty \le R\}.$$

Clearly, the subset D_R is closed, bounded, and convex. We will show that N satisfies the assumptions of Theorem 2.2. The proof will be given in three steps.

Step 1. N maps D_R into itself.

For each $x \in D_R$, by (H2) and (3.1) we have for each $t \in J$

$$\begin{aligned}\|(Nx)(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_1^e (\log t)^{\alpha-1} \left(\log \frac{e}{s}\right)^{\alpha-1} \|f(s, x(s))\| \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|f(s, x(s))\| \frac{ds}{s} + \int_1^t \|g(s, x(s))\| \frac{ds}{s} \\ &\le R\left(\frac{2}{\Gamma(\alpha+1)} \|p_f\|_\infty + \|p_g\|_\infty\right) \\ &\le R.\end{aligned}$$

Step 2. $N(D_R)$ is bounded and equicontinuous.

By Step 1, we have $N(D_R) = \{Nx : x \in D_R\} \subset D_R$. Thus, for each $x \in D_R$, we have $\|Nx\|_\infty \le R$, which means that ND_R is bounded. For the equicontinuity of $N(D_R)$. Let $t_1, t_2 \in J$, $t_1 < t_2$ and $x \in D_R$. Then

$$\begin{aligned}&\|(Nx)(t_2) - (Nx)(t_1)\| \\ &\le \frac{(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \|f(s, x(s))\| \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left| \left(\log \frac{t_1}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1} \right| \|f(s, x(s))\| \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \|f(s, x(s))\| ds + \int_{t_1}^{t_2} \|g(s, x(s))\| \frac{ds}{s} \\ &\le \frac{(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} p_f(s) \|x(s)\| \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1}\right) p_f(s) \|x(s)\| \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} p_f(s) \|x(s)\| ds + \int_{t_1}^{t_2} p_g(s) \|x(s)\| \frac{ds}{s} \\ &\le \frac{\|p_f\|_\infty R}{\Gamma(\alpha+1)} \left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1} + (\log t_2)^\alpha - (\log t_1)^\alpha\right) \\ &+ \|p_g\|_\infty R (\log t_2 - \log t_1).\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Step 3. N is continuous.

Let $\{x_n\}$ be sequence such that $x_n \to x$ in $C(J,E)$. Then, for each $t \in J$

$$\begin{aligned} & \|(Nx_n)(t) - (Nx)(t)\| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_1^e (\log t)^{\alpha-1} \left(\log \frac{e}{s}\right)^{\alpha-1} \|f(s,x_n(s)) - f(s,x(s))\| \frac{ds}{s} \\ & + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|f(s,x_n(s)) - f(s,x(s))\| \frac{ds}{s} \\ & + \int_1^t \|g(s,x_n(s)) - g(s,x(s))\| \frac{ds}{s}. \end{aligned}$$

Since f and g are Caratheodory functions, the Lebesgue dominated convergence theorem implies that

$$\|(Nx_n)(t) - (Nx)(t)\| \to 0 \text{ as } n \to \infty.$$

This shows that (Nx_n) converges pointwise to Nx on J. Moreover, the sequence (Nx_n) is equicontinuous by a similar proof of Step 2. Therefore (Nx_n) converges uniformly to Nx and hence N is continuous.

Now let V be a subset of D_R such that $V \subset \overline{conv}((NV) \cup \{0\})$. V is bounded and equicontinuous, and therefore the function $v \to v(t) = \mu(V(t))$ is continuous on J. By assumption (H3), Lemma 2.3 and the properties of the measure μ we have for each $t \in J$

$$\begin{aligned} v(t) & \leq \mu((NV)(t) \cup \{0\}) \leq \mu((NV)(t)) \\ & \leq \frac{1}{\Gamma(\alpha)} \int_1^e (\log t)^{\alpha-1} \left(\log \frac{e}{s}\right)^{\alpha-1} p_f(s)\, \mu(V(s)) \frac{ds}{s} \\ & + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_f(s)\, \mu(V(s)) \frac{ds}{s} + \int_1^t p_g(s)\, \mu(V(s)) \frac{ds}{s} \\ & \leq \|v\|_\infty \left(\frac{2}{\Gamma(\alpha+1)} \|p_f\|_\infty + \|p_g\|_\infty \right). \end{aligned}$$

This means that

$$\|v\|_\infty \left(1 - \left[\frac{2}{\Gamma(\alpha+1)} \|p_f\|_\infty + \|p_g\|_\infty \right] \right) \leq 0.$$

By (3.1), it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D_R. Applying now Theorem 2.2, we conclude that N has a fixed point, which is a solution of the problem (1.1). □

4. EXAMPLE

We consider the following Hadamard fractional differential equation

$$\begin{cases} D^{\frac{5}{4}} x(t) - \frac{1}{1+3\exp(t)} x(t) = D^{\frac{1}{4}} \frac{1}{3+2\exp(t^2)} x(t),\ t \in J = [1,e], \\ x(1) = 0,\ x(e) = \int_1^e \frac{1}{3+2\exp(s^2)} x(s)\, ds. \end{cases} \tag{4.1}$$

Let

$$E = l^1 = \left\{ x = (x_1, x_2, ..., x_n, ...) : \sum_{n=1}^{\infty} |x_n| < \infty \right\},$$

equipped with the norm

$$\|x\|_E = \sum_{n=1}^{\infty} |x_n| .$$

Set

$$x = (x_1, x_2, ..., x_n, ...) , \; f = (f_1, f_2, ..., f_n, ...) , \; g = (g_1, g_2, ..., g_n, ...) ,$$

and

$$f_n (t, x_n) = \frac{1}{1 + 3 \exp (t)} x_n, \; t \in J,$$
$$g_n (t, x_n) = \frac{1}{3 + 2 \exp (t^2)} x_n, \; t \in J.$$

For each x_n and $t \in J$, we have

$$|f_n (t, x_n)| \leq \frac{1}{1 + 3 \exp (t)} |x_n| , \tag{4.2}$$

and

$$|g_n (t, x_n)| \leq \frac{1}{3 + 2 \exp (t^2)} |x_n| . \tag{4.3}$$

Hence conditions (H1) and (H2) are satisfied with $p_f (t) = \frac{1}{1+3 \exp(t)}$ and $p_g (t) = \frac{1}{3+2 \exp(t^2)}$. By (4.2) and (4.3), for any bounded set $B \subset l^1$, we have

$$\mu (f (t, B)) \leq \frac{1}{1 + 3 \exp (t)} \mu (B) \text{ for each } t \in J,$$
$$\mu (g (t, B)) \leq \frac{1}{3 + 2 \exp (t^2)} \mu (B) \text{ for each } t \in J.$$

Hence (H3) is satisfied. The condition

$$\frac{2}{\Gamma (\alpha + 1)} \|p_f\|_\infty + \|p_g\|_\infty \simeq 0.64 < 1,$$

is satisfied. Consequently, Theorem 3.2 implies that the problem (4.1) has a solution defined on J.

5. Conclusion

In this paper, we provided the existence of solutions of Hadamard fractional differential equations in a Banach space. The main tool of this work is the measure of noncompactness and the Mönch fixed point theorem. However, by introducing a new fixed mapping, we get new existence conditions. The obtained results have a contribution to the related literature, and they extend the results in [21].

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References

[1] M. S. Abdo, Further results on the existence of solutions for generalized fractional quadratic functional integral equations, J. Math. Anal. Model., 1(1) (2020), 33–46.

[2] M. S. Abdo, S. K. Panchal, An existence result for fractional integro-differential equations on Banach space, Journal of Mathematical Extension, 13(3) (2019), 19–33.

[3] M. S. Abdo, S. T. M. Thabet, B. Ahmad, The existence and Ulam–Hyers stability results for ψ-Hilfer fractional integrodifferential equations, J. Pseudo-Differ. Oper. Appl., 11 (2020), 1757–1780.
[4] R. P. Agarwal, M. Meehan, and D. O'Regan, Fixed Point Theory and Applications, Vol. 141 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, UK, 2001.
[5] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii, Measures of Noncompactness and Condensing Operators, Oper. Theory Adv. Appl. 55, Birkhäuser, Basel, 1992.
[6] A. Ardjouni, Positive solutions for nonlinear Hadamard fractional differential equations with integral boundary conditions, AIMS Mathematics, 4(4) (2019), 1101–1113.
[7] A. Ardjouni, A. Djoudi, Positive solutions for first-order nonlinear Caputo-Hadamard fractional relaxation differential equations, Kragujevac Journal of Mathematics, 45(6) (2021), 897–908.
[8] A. Ardjouni, A. Djoudi, Initial-value problems for nonlinear hybrid implicit Caputo fractional differential equations, Malaya Journal of Matematik, 7(2) (2019), 314–317.
[9] A. Ardjouni, A. Djoudi, Approximating solutions of nonlinear hybrid Caputo fractional integro-differential equations via Dhage iteration principle, Ural Mathematical Journal, 5(1) 2019, 3–12.
[10] A. Ardjouni, A. Djoudi, Existence and uniqueness of positive solutions for first-order nonlinear Liouville-Caputo fractional differential equations, São Paulo J. Math. Sci., 14 (2020), 381–390.
[11] A. Ardjouni, A. Lachouri, A. Djoudi, Existence and uniqueness results for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations, Open Journal of Mathematical Analysis, 3(2) (2019), 106–111.
[12] J. Banaś, K. Goebel, Measures of Noncompactness in Banach Spaces, Lect. Notes Pure Appl. Math. 60, Marcel Dekker, New York, 1980.
[13] M. Benchohra, A. Cabada and D. Seba, An existence result for nonlinear fractional differential equations on Banach spaces, Bound. Value. Probl., 2009(1) (2009), 1–11.
[14] M. Benchohra, J. Henderson, D. Seba, Measure of noncompactness and fractional differential equations in Banach spaces, Commun. Appl. Anal., 12(4) (2008), 419–427.
[15] M. Benchohra, J. Henderson, D. Seba, Boundary value problems for fractional differential inclusions in Banach spaces, Fract. Differ. Calc., 2(1) (2012), 99–108.
[16] M. Benchohra, G. M. N'Guérékata, D. Seba, Measure of noncompactness and nondensely defined semilinear functional differential equations with fractional order, Cubo, 12(3) (2010), 35–48.
[17] C. Derbazi, Z. Baitiche, M. S. Abdo, T. Abdeljawad, Qualitative analysis of fractional relaxation equation and coupled system with Ψ-Caputo fractional derivative in Banach spaces, AIMS Mathematics, 6(3) (2021), 2486–2509.
[18] K. Diethelm, The analysis of Fractional Differential equations, Lecture Notes in Mathematics, Springer-verlag, Berlin, Heidelberg, 2010.
[19] A. Hamrouni, S. Beloul, A Aissaoui, Existence of solutions for boundary value problems of fractional integro-differential equations with integral boundary conditions on Banach spaces, Nonlinear Studies, 26(3) (2019), 693–701.
[20] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B. V., Amsterdam, 2006.
[21] A. Lachouri, A. Ardjouni, A. Djoudi, Existence results for integral boundary value problems of fractional differential equations with Two nonlinear terms in Banach spaces, Bull. Int. Math. Virtual Inst., 11(1) (2021), 159–168.
[22] A. Lachouri, A. Ardjouni, A. Djoudi, Positive solutions of a fractional integro-differential equation with integral boundary conditions, Communications in Optimization Theory, 2020 (2020), 1–9.
[23] L. A. Palve, M. S. Abdo, S. K. Panchal, Some existence and stability results of Hilfer-Hadamard fractional implicit differential fractional equation in a weighted space, arXiv:1910.08369v1.
[24] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[25] D. Seba, Nonlinear fractional differential inclusion with nonlocal fractional integro differential boundary conditions in Banach Spaces, Math. Bohem., 142(3) (2017), 1–13.
[26] D. Seba, H, Rebai, J, Henderson, Existence result for nonlinear fractional differential equations with nonlocal fractional integro differential boundary conditions in Banach spaces, Georgian Math. J., 2019 (2019), 1–7.
[27] S. A. Szufla, On the application of measure of noncompactness to existence theorems, Rend. Semin. Mat. Univ. Padova, 75 (1986), 1–14.
[28] H. A. Wahash, M. S. Abdo, S. K. Panchal, S. P. Bhairat, Existence of solution for Hilfer fractional differential problem with nonlocal boundary condition, arXiv:1909.13679v1.
[29] H. A. Wahash, S. K. Panchal, Positive solutions for generalized two-term fractional differential equations with integral boundary conditions, J. Math. Anal. Model., 1(1) (2020), 47–63.

[30] H. A. Wahash, S. K. Panchal, Positive solutions for generalized Caputo fractional differential equations using lower and upper solutions method, J. Frac. Calc. Nonlinear Sys., 1(1) (2020), 1–12.
[31] F. Wang, Existence and uniqueness of solutions for a nonlinear fractional differential equation, J. Appl. Math. Comput., 39(1-2) (2012), 53–67.
[32] M. Xu, S. Sun, Positivity for integral boundary value problems of fractional differential equations with two nonlinear terms, J. Appl. Math. Comput., 59(12) (2019), 271–283.
[33] Y. Zhou, J. R. Wang, L. Zhang, Basic Theory of Fractional Differential Equations, Second edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.

ABDELOUAHEB ARDJOUNI
DEPARTMENT OF MATHEMATICS AND INFORMATICS, UNIVERSITY OF SOUK AHRAS, P.O. BOX 1553, SOUK AHRAS, 41000, ALGERIA
APPLIED MATHEMATICS LAB, FACULTY OF SCIENCES, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ANNABA, P.O. BOX 12, ANNABA 23000, ALGERIA
Email address: abd_ardjouni@yahoo.fr