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THE THIRD ISOMORPHISM THEOREM FOR (CO-ORDERED) Γ-SEMIGROUPS WITH APARTNESS

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ABSTRACT. The notion of Γ-semigroups has been introduced by M. K. Sen and N. K. Saha. The concept of (co-ordered) Γ-semigroups with apartness in Bishop's constructive algebra was introduced by this author. Many classical notions and processes of semigroups and Γ-semigroups have been extended to (co-ordered) Γ-semigroups with apartness such as ideals, filters and the first theorem of isomorphism of this class of algebraic structures. In this paper, as a continuation of earlier research, the author designs a form of the third isomorphism theorems for Γ-semigroups and co-ordered Γ-semigroups with apartness which does not have its counterpart in the classical Γ-semigroup theory.

1. INTRODUCTION

The principle-logical framework of this article is Bishop constructive mathematics ([1, 2, 8]) which implies intuitionistic logic ([16]).

Within this framework the author is interested in Γ-semigroups with apartness as a continuation of his research [9, 11, 12, 13]. The concept of the Γ-semigroup with apartness was introduced in the article [9]. Semilattice congruences in Γ-semigroup with apartness were the focus of the paper [11] while the article [12] analyzes some substructures of co-ordered Γ-semigroup with apartnss such as co-filters. In article [13], two forms of the first theorem on isomorphism between (co-ordered) Γ-semigroups with apartness are considered.

In this report, as a continuation of previous research, the author presents a form of the third theorem on isomorphism between (co-ordered) Γ-semigroups with apartness which does not have its counterpart in the classical Γ-semigroup theory: Theorem 3.2 for Γ-semigroup with apartness and Theorem 3.4 for co-ordered Γ-semigroup with apartness.

The concept of co-congruences on Γ-semigroup with apartness was introduced and analyzed in the author's article [9]. The presence of a Γ -cocongruence q on a (co-ordered) Γ-semigroup with apartness $(S, =, \neq, w)$ it allows the construction of a (co-ordered) Γ - semigroup with apartness $[S : q] = \{xq : x \in S\}$. This type of (co-ordered) Γ-semigroups has no counterpart in the classical theory of Γ-semigroups. Let q_1 and q_2 be co-congruences on a Γ-semigroup with apartness S such that $q_2 \subseteq q_1$. Theorem 3.2

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shows the relationship between Γ-semigroups $[S : q_1]$ and $[S : q_2]$. Theorem 3.4 describes an analogous relationship between Γ-semigroups $[S : q_1]$ and $[S : q_2]$ if Γ-semigroup with apartness S is ordered with respect to two co-orders relations.

2. Preliminaries

The concept of Γ-semigroup was developed in the papers [14, 15]. Articles [3, 4] study isomorphisms on (ordered) Γ-semigroups.

2.1. **Γ-semigroup with apartness.** To explain the notions and notations used in this article, which but not previously described, we instruct the reader to look at the articles [3, 4, 6, 7, 14, 15]. Here we will introduce some specific notions and substructures of this semigroups that appear only in the **Bish** version by reference to the author's reports [9, 10, 11, 12, 13].

Definition 2.1. ([9], Definition 2.1) Let $(S, =_S, \neq_S)$ and $(\Gamma, =_\Gamma, \neq_\Gamma)$ be two non-empty sets with apartness. Then S is called a Γ-*semigroup with apartness* if there exist a strongly extensional total mapping

$$w_S : S \times \Gamma \times S \ni (x, a, y) \longmapsto w_S(x, a, y) := xay \in S$$

satisfying the condition

$$(\forall x, y, z \in S)(\forall a, b \in \Gamma)((xay)bz =_S xa(ybz)).$$

We recognize immediately that the following implication

$$(\forall x, y, u, v \in S)(\forall a, b \in \Gamma)(xay \neq_S ubv \implies (x \neq_S u \vee a \neq_\Gamma b \vee y \neq_S v))$$

is valid, because w_S is a strongly extensional function.

Example 2.2. Let $\mathbb{N}$ be a semiring of natural numbers, $\mathbb{R}$ be the filed of real number, where the apartness is determined as follows

$$(\forall x, y \in \mathbb{R})(x \neq y \iff (\exists k \in \mathbb{N})(|x - y| > \frac{1}{k})),$$

$S := [0, 1] \subseteq \mathbb{R}$ and $\Gamma := \{\frac{1}{n} : n \in \mathbb{N}\}$. Then S is a commutative Γ-semigroup under the usual multiplication.

Example 2.3. Let S be the set $M_{2\times3}(\mathbb{R})$ of all 2×3 matrices over the set of real numbers $\mathbb{R}$ and Γ be the set $M_{3\times2}(\mathbb{R})$ of all 3×2 matrices over $\mathbb{R}$. Apartness '$\neq$' is defined in $M_{2\times3}(\mathbb{R})$ in the standard way

$$(\forall A, B \in M_{2\times3}(\mathbb{R}))(A \neq B \iff (\exists (i, j) \in \{1, 2\} \times \{1, 2, 3\})(a_{ij} \neq b_{ij})).$$

The apartness relation in $M_{3\times2}(\mathbb{R})$ is defined analogously. Define $A\alpha B$ = usual matrix product of A, α, B for all $A, B \in S$ and for all $\alpha \in \Gamma$. Then S is a Γ-semigroup with apartness. Note that S is not a semigroup.

Definition 2.4. ([9], Definition 2.2) Let S be a Γ-semigroup with apartness. A subset T of S is said to be a Γ-*cosubsemigroup* of S if the following holds

$$(\forall x, y \in S)(\forall a \in \Gamma)(xay \in T \implies (x \in T \vee y \in T)).$$

We will assume that the empty set $\emptyset$ is a Γ-cosubsemigroup of a Γ-semigroup S by definition.

2.2. **Co-ordered Γ-semigroup with apartness.** The relation α is said to be a co-order on the set $(X, =_X, \neq_X)$ with apartness if it is consistent $\alpha \subseteq \neq_X$, co-transitive $\alpha \subseteq \alpha * \alpha$, i.e.

$$(\forall x, y, z \in X)((x, z) \in \alpha \implies ((x, y) \in \alpha \vee (y, z) \in \alpha))$$

and linear in the following sense: $\neq_X \subseteq \alpha \cup \alpha^{-1}$. α is said to be a co-quasiorder on X if it is a consistent and co-transitive relation on X.

In order to avoid misunderstanding the notations used, we remind yourself that equality and apartness are determined on the product $S \times T$ in the following way

$$(\forall x, y \in S)(\forall u, v \in T)((x, u) = (y, v) \iff (x =_S y \wedge u =_T v))$$

and

$$(\forall x, y \in S)(\forall u, v \in T)((x, u) \neq (y, v) \iff (x \neq_S y \vee u \neq_T v)).$$

A brief recapitulation of a number of algebraic structures ordered by co-quasiorder relation is presented in the review paper [10].

In the following definition we introduce the concept of co-order relations in Γ-semigroup with apartnesst.

Definition 2.5. ([11], Definition 3.1) Let S be a Γ-semigroup with apartness. A co-order relation $\nleqslant_S$ on S is *compatible* with the semigroup operations in S if the following holds

$$(\forall x, y, z \in S)(\forall a \in \Gamma)((xaz \nleqslant_S yaz \vee zax \nleqslant_S zay) \implies x \nleqslant_S y).$$

In this case it is said that S is an *ordered* Γ-semigroup *under co-order* $\nleqslant_S$ or it is *co-ordered* Γ-semigroup.

If we speak the language of classical algebra, then the relation $\nleqslant_S$ is compatible with the operation w_S in S if this operation is cancellative with respect to the co-order.

Example 2.6. Let $M = \{a, b, c\}$, where is it $a \neq b$, $b \neq c$ and $a \neq c$, and $\Gamma = \{\gamma\}$ with the internal operation w_M defined as in [5], Example 1.11. Then M is a commutative Γ-semigroup. Define a relation $\nleqslant$ on M as follows $\nleqslant := \{(b, a), (b, c), (c, a), (c, b)\}$. Then M is a co-ordered set. It can be verified that M is a ordered Γ-semigroup under the co-order relation $\nleqslant$.

Example 2.7. Let $\mathbb{N}$ be additive semigroup of natural numbers. If we put $S := (\mathbb{N}, =, \neq, +, \nleqslant)$, $\Gamma := \{5\}$, where an apartness $\neq$ is determined by $x \neq y \iff \neg(x = y)$, and $w(x, 5, y) = x + 5 + y \in \mathbb{N}$, then S is ordered Γ-semigroup under the co-order $\nleqslant$ determined by $x \nleqslant y \iff \neg(x \leqslant y)$.

Example 2.8. Let S be a set of all reverse isotone se-mapping from an ordered set $(P, =, \neq)$ under a co-order $\nleqslant_P$ into another ordered set $(Q, =, \neq)$ under the co-order $\nleqslant_Q$ and let Γ be set of all reverse isotone se-mapping from Q to P. In both cases, apartness is defined as follows $f \neq g \iff (\exists x)(f(x) \neq g(x))$. For $f, g \in S$ and $\alpha \in \Gamma$ put $f\alpha g = g \circ \alpha \circ f$, where $' \circ '$ be mark for standard composition between relations. Then S is a Γ-semigroup. A co-order $' \nleqslant '$ relation on S can defined by $f \nleqslant g \iff (\exists x \in P)(f(x) \nleqslant_Q g(x))$. Then S is a co-ordered Γ-semigroup.

2.3. **Γ-cocongruences.** The notion of the co-equality relation in sets with apartness introduced and analyzed by this author (see, for example, [10]). The relation q is a co-equality on a set $(S, =_S, \neq_S)$ if it is a consistent, symmetric and co-transitive relation on S. A co-congruence on some algebraic structure $((S, =_S, \neq_S), \cdot)$ is a coequality relation on S which is compatible in one very specific sense with the internal operation in S. A reader can look at these specific features in the author review article [10]. A co-equality relation q on a semigroup with apartness S is a co-congruence in S if the following holds

$$(\forall x, y, u, v \in S)((xu, yv) \in q \implies ((x, y) \in q \vee (u, v)).$$

The concept of Γ-cocongruence on Γ-semigroups with apartness was introduced in [9] by the following definition

Definition 2.9. ([9], Definition 2.6) Let S be a Γ-semigroup with apartness. A co-equality relation $q \subseteq S \times S$ is called a Γ - *cocongruence* on S if the following holds

$$(xau, ybv) \in q \implies ((x, y) \in q \vee a \neq_\Gamma b \vee (u, v) \in q)$$

for any $x, y, u, v \in S$ and all $a, b \in \Gamma$.

Let q be a co-congruence on Γ-semigroup with apartness S. This tool it allows the construction of the congruence relation $q^\lhd$ on S compatible with q. The pair $(q^\lhd, q)$ it allows to design Γ-semigroup with apartness $S/(q^\lhd, q)$. In addition to this structure, the relation q it allows the design of Γ-semigroup with apartness $[S : q] := \{xq : x \in S\}$ which has no counterpart in the classical Γ-semigroup theory.

In this paper, the focus is on this structure and in what follows, some of the properties of this last algebraic structure are analyzed.

2.4. **Homomorphism between Γ-semigroups.** In this subsection, the determination of the concept of Γ-homomorphism between Γ-semigroups with apartness is taken from article [9].

Definition 2.10. ([9], Definition 2.7) Let $(S, =_S, \neq_S)$ is a Γ-semigroup and $(T, =_T, \neq_T)$ a Λ - semigroups with apartness. A pair (h, φ) of strongly extensional functions $h : S \longrightarrow T$ and $\varphi : \Gamma \longrightarrow \Lambda$ is called a *se-homomorphism* from Γ-semigroup S to Λ-semigroup T if the following holds

$$(\forall x, y \in S)(\forall a \in \Gamma)((h, \varphi)(xay) =_T h(x)\varphi(a)h(y)).$$

The following can be verified without difficulty.

Lemma 2.1. *Let (h, φ) be a se-homomorphism from Γ-semigroup with apartness S to a Λ-semigroup with apartness R. Then holds*

$$(h, \varphi) \circ w_S = w_T \circ (h, \varphi, h)$$

where (h, φ, h) is understood as follows:

$$(\forall x, y \in S)(\forall a \in \Gamma)((h, \varphi, h)(x, a, y) := (h(x), \varphi(a), h(y))).$$

Of course, the specificity of this determination is in the requirement that the functions $h : S \longrightarrow T$ and $\varphi : \Gamma \longrightarrow \Lambda$ must be strongly extensional functions. It is easily verified that (h, φ) is a correctly determined strongly extensive function.

Also, it is easy to see that:

Proposition 2.2. *Let $(h, \varphi) : S \longrightarrow T$ be a se-homomorphism. Then the relation*

$$q := Coker(h, \varphi) := \{(x, y) : (h, \varphi)(x) \neq_T (h, \varphi)(y)\}$$

is a Γ-cocongruence on S.

3. The Main Results

In this section, which is the central part of this paper, we are talking about 'the Third theorem on isomorphism between Γ-semigroups with apartness' and 'the Third theorem on isomorphism between ordered Γ-semigroup with apartness under a co-order. The specificity of this presentation is that the forms of this theorem that do not have their counterparts in the classical Γ-semigroup theory.

3.1. **The case of Γ-semigroup with apartness.** Let q_1 and q_2 be co-congruences on a Γ-semigroup with apartness $(S, =_S, \neq_S, w_S)$ such that $q_2 \subseteq q_1$. We can construct Γ-semigroups with apartness $([S : q_1], =_1, \neq_1, w_{[S:q_1]})$ and $([S : q_2], =_2, \neq_2, w_{[S:q_2]})$. In the following theorem we will give a construction of relation $[q_2 : q_1]$ on $[S : q_1]$ by relations q_1 and q_2.

Theorem 3.1. *Let q_1 and q_2 be co-congruences on a Γ-semigroup with apartness S such that $q_2 \subseteq q_1$. Then the relation $[q_2 : q_1] \subseteq [S : q_1] \times [S : q_1]$, defined by*

$$(\forall xq_1, yq_1 \in [S : q_1])((xq_1, yq_1) \in [q_2 : q_1] \iff (x, y) \in q_2),$$

is a co-congruence on $[S : q_1]$.

Proof. We will first show that $[q_2 : q_1]$ is a co-equality relation on $[S : q_1]$. Let $x, y, z \in S$ be elements such that $(xq_1, zq_1) \in [q_2 : q_1]$. Then

(i) $(xq_1, zq_1) \in [q_2 : q_1] \iff (x, z) \in q_2$
$\implies ((x, y) \in q_2 \vee (y, z) \in q_2)$
$\implies ((xq_1, yq_1) \in [q_2 : q_1] \vee (yq_1, zq_1) \in [q_2 : q_1])$.

(ii) $(xq_1, yq_1) \in [q_2 : q_1] \iff (x, y) \in q_2$
$\iff (y, x) \in q_2$
$\iff (yq_1, xq_1) \in [q_2 : q_1]$.

(iii) $(xq_1, yq_1) \in [q_2 : q_1] \implies (x, y) \in q_2 \subseteq q_1$
$\iff xq_1 \neq_1 yq_1$.

Second, let us check that $[q_2 : q_1]$ is a co-congruence on $[S : q_1]$. Take $x, y, u, v \in S$ and $a, b \in \Gamma$ such that $((xq_1)a(yq_1), (uq_1)b(vq_1)) \in [q_2 : q_1]$. Then

$$\begin{aligned}((xay)q_1, (ubv)q_1) \in [q_2 : q_1] &\iff (xay, ubv) \in q_2\\ &\implies (x, u) \in q_2 \subseteq q_1 \vee a \neq_\Gamma b \vee (y, v) \in q_2 \subseteq q_1\\ &\implies xq_1 \neq_1 yq_1 \vee a \neq_\Gamma b \vee yq_1 \neq_1 vq_1.\end{aligned}$$ □

We can ([13], Theorem 3.3) design Γ-semigroup

$$[[S; q_1] : [q_2 : q_1]] := \{(xq_1)[q_2 : q_1] : xq_1 \in [S : q_1]\}$$

with

$$(\forall x, y \in S)((xq_1)[q_2 : q_1] =_3 (yq_1)[q_2 : q_1] \iff (xq_1, yq_1) \lhd [q_2 : q_1])$$

and

$$(\forall x, y \in S)((xq_1)[q_2 : q_1] \neq_3 (yq_1)[q_2 : q_1] \iff (xq_1, yq_1) \in [q_2 : q_1]),$$

where the internal operation

$w_3 := w_{[[S:q_1]:[q_2:q_1]]} : [[S; q_1 : [q_2 : q_1]] \times \Gamma \times [[S : q_1] : [q_2 : q_1]] \longrightarrow [[S : q_1] : [q_2 : q_1]]$

is determined as follows

$$w_3((xq_1)[q_2 : q_1], a, (yq_1)[q_2 : q_1]) := ((xay)q_1)[q_2 : q_1].$$

Also, according to Theorem 3.3 in [13], there are unique se-epimorphisms

$$\vartheta_1 : S \longrightarrow [S : q_1], \vartheta_2 : S \longrightarrow [S : q_2] \text{ and } \vartheta_{21} : [S : q_1] \longrightarrow [[S : q_1] : [q_2 : q_1]].$$

In order to be able to design the third isomorphism theorem for this class of γ-semigroups with apartness we need the following lemma

One form of the third isomorphism theorem between Γ-semigroups with apartness that does not have its counterpart in the classical theory of Γ-semigroups can now be designed.

Theorem 3.2. *Let q_1 and q_2 be co-congruences on a Γ-semigroup with apartness S such that $q_2 \subseteq q_1$. Then there is a unique injective, embedding and surjective se-homomorphism $\beta : [S : q_1] : [q_2 : q_1]] \longrightarrow [S : q_2]$ such that $\vartheta_2 = \beta \circ \vartheta_{21} \circ \vartheta_1$.*

Proof. Define $\beta : [[S : q_1] : [q_2 : q_1]] \longrightarrow [S : q_2]$ by

$$(\forall x[q_2 : q_1] \in [[S : q_1] : [q_2 : q_1]])(\beta((xq_1)[q_2 : q_1]) := xq_2).$$

(a) First, let's show that β is well-defined se-mapping.

Let $x, y, u, v \in S$ be such that $(xq_1)[q_2 : q_1] =_3 (yq_1)[q_2 : q_1]$ and $(uq_1, vq_1) \in [q_2 : q_1]$. Then $(xq_1, yq_1) \lhd [q_2 : q_1]$ and $(u, v) \in q_2$. Thus

$$\begin{aligned}
&(uq_1, vq_1) \in [q_2 : q_1] \implies \\
&\qquad (uq_1, xq_1) \in [q_2 : q_1] \vee (xq_1, yq_1) \in [q_2 : q_1] \vee (yq_1, vq_1) \in [q_2 : q_1] \\
&\qquad \implies (u, x) \in q_2 \vee (y, v) \in q_2 \\
&\qquad \implies u \neq_S x \vee y \neq_S v \\
&\qquad \implies (x, y) \neq (u, v) \in q_2.
\end{aligned}$$

This means $(x, y) \lhd q_2$. Thus

$$\beta((xq_1)[q_2 : q_1]) := xq_2 =_2 yq_2 := \beta((yq_1)[q_2 : q_1]).$$

On the other hand, if

$$\beta((xq_1)[q_2 : q_1]) \neq_2 \beta((yq_1)[q_2 : q_1]),$$

we have $xq_2 \neq_2 yq_2$. Then $(x, y) \in q_2$. Thus $(xq_1, yq_1) \in [q_2 : q_1]$. So,

$$(xq_1)[q_2 : q_1] \neq_3 (yq_1)[q_2 : q_1].$$

(b) Second, it should be shown that β is an injective mapping. Let $x, y, u, v \in S$ be such that

$$\beta((xq_1)[q_2 : q_1]) =_2 \beta((yq_1)[q2 : q_1])$$

and $(uq_1, vq_1) \in [q_2 : q_1]$. This means $xq_2 =_2 yq_2$ and $(u, v) \in q_2$. Then $(x, y) \lhd q_2$. Further on, from $(u, v) \in q_2$ we have

$$\begin{aligned}
(u, v) \in q_2 &\implies (u, x) \in q_2 \subseteq q_1 \vee (x, y) \in q_2 \vee (y, v) \in q_2 \subseteq q_1 \\
&\implies uq_1 \neq_1 xq_1 \vee yq_1 \neq vq_1 \\
&\implies (xq_1, yq_1) \neq (uq_1, vq_1) \in [q_2 : q_1].
\end{aligned}$$

This means $(xq_1, yq_1) \lhd [q_2 : q_1]$. Hence

$$(xq_1)[q_2 : q_1] =_3 (yq_1)[q_2 : q_1].$$

(c) Let us prove that β is an embedding. Let $x, y \in S$ be elements such that $(xq_1)[q_2 : q_1] \neq_3 (yq_1)[q_2 : q_1]$. Then $(xq_1, yq_1) \in [q_2 : q_1]$. So, $(x, y) \in q_2$. Hence $xq_2 \neq_2 yq_2$. This means $\beta((xq_1)[q_2 : q_1]) \neq_2 \beta((yq_1)[q_2 : q_1])$.

(d) Since it is obvious that β is surjective, it remains to show that (β, i) is a homomorphism of Γ-semigroups. For $x, y \in S$ and $a \in \Gamma$, we have

$$\begin{aligned}
&(\beta, i)(((xq_1)[q_2 : q_1])\, a\, ((yq_1)[q_2 : q_1])) =_2 \\
&(\beta, i)(w_3((xq_1)[q_2 : q_1],\, a,\, (yq_1)[q_2 : q_1])) =_2 (\beta, i)(((xay)q_1)[q_2 : q_1]) \\
&=_2 (xay)q_2 =_2 w_2(xq_2,\, a,\, yq_2) =_2 w_2(\beta((xq_1)[q_2 : q_1]),\, a,\, \beta((yq_1)[q_2 : q_1]))
\end{aligned}$$

$=_2 (\beta((xq_1)[q_2 : q_1))a(\beta((yq_1)[q_2 : q_1])).$

(e) Finally, we show that the required equality is valid. For arbitrary $x \in S$ we have

$$\begin{aligned}\vartheta_2(x) &:= xq_2 =_2 \beta((xq_1)[q_2 : q_1]) \\ &=_2 \beta(\vartheta_{21}(xq_1)) =_2 \beta(\vartheta_{21}(\vartheta_1(x))) \\ &=_2 (\beta \circ \vartheta_{21} \circ \vartheta_1)(x).\end{aligned}$$

□

3.2. **The case of co-ordered Γ-semigroup with apartness.** Let us consider ordered Γ-semigroup with apartness $(S, =_S, \neq_S, \nleqslant_S)$ under a co-order $\nleqslant_S$ and let σ and τ be co-quasiorder relation on S such that $\sigma \subseteq \tau \subseteq \nleqslant_S$ and suppose that σ and τ satisfy the condition of Definition 2.5:

$$(\forall x, y, z \in S)(\forall a \in \Gamma))(((xaz, yaz) \in \sigma \vee (zax, zay) \in \sigma) \implies (x, y) \in \sigma),$$

$$(\forall x, y, z \in S)(\forall a \in \Gamma))(((xaz, yaz) \in \tau \vee (zax, zay) \in \tau) \implies (x, y) \in \tau).$$

It is known that co-congruences $q_\sigma = \sigma \cup \sigma^{-1}$ and $q_\tau = \tau \cup \tau^{-1}$ on S such that $q_\sigma \subseteq q_\tau$ can be designed. Further, by Theorem 4.3 in [13], this allows us to construct Γ-semigroups with apartness $([S : q_\sigma], =_\sigma, \neq_\sigma, w_\sigma)$ and $([S : q_\tau], =_\tau, \neq_\tau, w_\tau)$ which are ordered by co-order relations $\nleqslant_\sigma$ and $\nleqslant_\tau$ respectfully as follows

$$(\forall x, y \in S)(xq_\sigma \nleqslant_\sigma yq_\sigma \iff (x, y) \in \sigma)$$

and

$$(\forall x, y \in S)(xq_\tau \nleqslant_\tau yq_\tau \iff (x, y) \in \tau).$$

Let us define the relation $[\sigma : \tau]$ on co-ordered Γ-semigroup with apartness $[S : \tau]$ as follows

$$(\forall x, y \in S)((xq_\tau, yq_\tau) \in [\sigma : \tau] \iff (x, y) \in \sigma).$$

Theorem 3.3. *Let σ and τ be co-quasiorder on a co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, \nleqslant_S)$ with the internal operation $w_S : S \times \Gamma \times S \longrightarrow S$ such that $\sigma \subseteq \tau \subseteq \nleqslant_S$. Then $[\sigma : \tau]$ is a co-quasiorder relation on $[S : q_\tau]$ compatible with the internal operation w_τ.*

Proof. Ler $x, y, z \in S$ be arbitrary elements. Then:

$$\begin{aligned}(xq_\tau, yq_\tau) \in [\sigma : \tau] &\implies (x, y) \in \sigma \subseteq \tau \subseteq q_\tau \\ &\implies xq_\tau \neq_\tau yq_\tau; \\ (xq_\tau, zq_\tau) \in [\sigma : \tau] &\iff (x, z) \in \sigma \\ &\implies (x, y) \in \sigma \vee (y, z) \in \sigma \\ &\implies (xq_\tau, yq_\tau) \in [\sigma : \tau] \vee (yq_\tau, zq_\tau) \in [\sigma : \tau].\end{aligned}$$

Let us show that this relation is compatible with the internal operation w_τ in $[S : q_\tau]$. For $x, y, z \in S$ we have

$$\begin{aligned}((xaz)q_\tau, (yaz)q_\tau) \in [\sigma : \tau] &\iff (xaz, yaz) \in \sigma \\ &\implies (x, y) \in \sigma \\ &\iff (xq_\tau, yq_\tau) \in [\sigma : \tau].\end{aligned}$$

The second implication can be proved in an analogous way. □

For ease of writing, let's put

$$q := q_{[\sigma:\tau]} = [\sigma : \tau] \cup [\sigma : \tau]^{-1}.$$

Without major difficulties it can be verified that q is a co-congruence on the co-ordered Γ-semigroup with apartness $([S, q_\tau], =_\tau, \neq_\tau, \nleqslant_\tau)$ compatible with its internal operation

w_τ. According to the previous theorem, Theorem 3.3, Γ-semigroup with apartness

$$([[S:q_\tau]:q],=_3,\neq_3,\not\leq_3,w_3)$$

can be designed, where is

$$(\forall x,y\in S)((xq_\tau)q=_3(yq_\tau)q \iff (xq_\tau,yq_\tau)\lhd q),$$

$$(\forall x,y\in S)((xq_\tau)q\neq_3(yq_\tau)q \iff (xq_\tau,yq_\tau)\in q).$$

The internal operation w_3 in $[[S;q_\tau]:q]$ is determined as follows

$$w_3((xq_\tau)q,\, a\,(yq_\tau)q):=(xq_\tau)q\cdot a\cdot(yq_\tau)q:=((xay)q_\tau)q$$

for any $(xq_\tau)q,(yq_\tau)q\in[[S:q_\tau]:q]$ and $a\in\Gamma$.
The co-order relation $\not\leq_3$ in $[[S:q_\tau]:q]$ is determined as follows

$$(\forall x,y\in S)((xq_\tau)q\not\leq_3(yq_\tau)q \iff (xq_\tau,yq_\tau)\in[\sigma:\tau]).$$

We can now design and prove the following theorem which we recognize as 'The third isomorphism theorem between co-ordered Γ-semigroups with apartness'. Of course, this form of this theorem does not have its counterpart in the classical Γ-semigroup theory.

Theorem 3.4. *Let σ and τ be co-quasiorder relations on co-ordered Γ-semigro- up with apartness $(S,=_S,\neq_S,\not\leqslant_S,w_S)$ such that $\sigma\subseteq\tau\subseteq\not\leqslant_S$. Then there is a unique injective, embedding and surjective se-homomorphism*

$$\gamma:[[S;q_\tau]:q]\longrightarrow[S;q_\sigma].$$

Proof. Let us define γ by

$$(\forall(xq_\tau)q\in[[S:q_\tau]:q])(\gamma((xq_\tau)q):=xq_\sigma).$$

First, let us show that γ is a well-defined mapping. Assume $x,y,u,v\in S$ are such that $(xq_\tau)q=_3(yq_\tau)q$ and $(u,v)\in q_\sigma$. then $(xq_\tau,yq_\tau)\lhd q$. On the other hand, from $(u,v)\in q_\sigma$ we get $(u,x)\in q_\sigma\vee(x,y)\in q_\sigma\vee(y,v)\in q_\sigma$. If we assume that $(x,y)\in q_\sigma$ is valid, then we would have the following $(x,y)\in\sigma$ or $(y,x)\in\sigma$. It would follow from here

$$(x,y)\in\sigma\vee(y,x)\in\sigma\implies(xq_\tau,yq_\tau)\in[\sigma:\tau]\subseteq q\vee(yq_t,xq_t)\in[\sigma:\tau]\subseteq q$$

which would contradict the hypothesis $(xq_\tau,yq_\tau)\lhd q$. So, it has to be $(u,x)\in q_\sigma$ or $(y,v)\in q_\sigma$. Thus $x\neq_S u$ or $y\neq_S v$. Therefore, $(x,y)\neq(u,v)\in q_\sigma$. This means $(x,y)\lhd q_\sigma$. Hence

$$\gamma((xq_\tau)q):=xq_\sigma=_\sigma yq_\sigma:=\gamma((yq_\tau)q).$$

Second, let us show that γ is a se-mapping. Let $x,y\in S$ be such that

$$xq_\sigma=\gamma((xq_\tau)q)\neq_3\gamma((yq_\tau)q)=yq_\sigma.$$

Then $(x,y)\in q_\sigma=\sigma\cup\sigma^{-1}$. Thus $((xq_\tau,yq_\tau)\in[\sigma:\tau]\cup[\sigma:\tau]^{-1}=q$. Hence

$$(xq_\tau)q\neq_3(yq_\tau)q.$$

Let $x,y,u,v\in S$ be such that $xq_\sigma=_\sigma yq_\sigma$ and $(uq_\tau,vq_\tau)\in q$. Then $(x,y)\lhd q_\sigma=\sigma\cup\sigma^{-1}$ and $(uq_\tau,vq_\tau)\in q$. Thus $(uq_\tau,xq_\tau)\in q$ or $(xq_\tau,yq_\tau)\in q$ or $(yq_\tau,vq_{tau})\in q$. The option $(xq_\tau,yq_\tau)\in q$ gives $(x,y)\in\sigma\cup\sigma^{-1}$ which is in contradiction with the hypothesis. So, it has to be $(uq_\tau,xq_\tau)\in q$ or $(yq_\tau,vq_\tau)\in q$. Thus $xq_\tau\neq_\tau uq_\tau$ or $yq_\tau\neq_\tau vq_\tau$. Hence $(xq_\tau,yq_\tau)\neq(uq_\tau,vq_\tau)\in q$. This means $(xq_\tau,yq_\tau)\lhd q$, i.e. $(xq_\tau)q=_3(yq_\tau)q$. This shows that γ is an injective mapping.

It remains to show that γ is an embedding. Let $x, y \in S$ be such that $(xq_\tau)q \neq_3 (yq_\tau)q$. Then $(xq_\tau, yq_\tau) \in q = [\sigma : \tau] \cup [\sigma : \tau]^{-1}$. Thus $(x, y) \in \sigma \cup \sigma^{-1} = q_\sigma$. Hence $xq_\sigma \neq_\sigma yq_\sigma$.

Finally, it is obvious that γ is a surjective mapping. □

4. Final comments

'Why could this selected material be of interest to the mathematical public?" — It's a very pertinent question that someone can ask. A significant number of members of this community do not fully understand some of the claims made in this and similar papers, and often the concepts and claims in them are thought of as something that should not deserve the attention of that community. The following could serve as a kind of justification:

Within the chosen principled-logical environment, the notions and processes with them treated in this and similar articles are logically possible.

New techniques have been designed and they have proven successful for processing settings in such a chosen work environment.

This enriches mathematics, our understanding of structures in algebra, but also opens new perspectives in the perception of observed algebraic structures.

Last but not least, this report presents the algebraic concepts Γ-semigroups with apartness and processes with them such as se-isomorphisms between them, which readers cannot encounter in the classical algebra.

In research what follows, the author will try to complete the research on Γ-semigroups with apartness by establishing of dual of (ordered) ideals and their properties in such algebraic structures.

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