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MULTIVARIATE RIGHT SIDE CAPUTO FRACTIONAL TAYLOR FORMULA AND LANDAU INEQUALITIES

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ABSTRACT. Here we present a multivariate right side Caputo fractional Taylor's formula with fractional integral remainder. Based on this we give three multivariate right side Caputo fractional Landau's type inequalities. Their constants are precisely calculated and we give best upper bounds.

1. INTRODUCTION

We are inspired by the following two results. First from [2], we have the following nice result:

If $\Delta f = \sum_{i=1}^{d} \left(\frac{\partial}{\partial x_i}\right)^2 f$ is the Laplacian of a function defined on $\mathbb{R}^d$, then for $0 < k < 2n$,

$$\left\| \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f \right\| \leq C(n,k) \|f\|^{1-\left(\frac{k}{2n}\right)} \|\Delta^n f\|^{\frac{k}{2n}}, \tag{1}$$

where $\|\cdot\|$ denotes the $L_\infty(\mathbb{R}^d)$ norm. $\Delta^n f$ needs to be defined only in the distributional sense and the result is valid in many other Banach spaces such as L^p spaces. $C(n,k)$ is an existential constant and $\Delta^k = \Delta(\Delta^{k-1})$.

From [3], we have the following interesting result which is the sharp inequality

$$\|\Delta f\|_\infty \leq 2\sqrt{\frac{d}{(d+2)}}\sqrt{\|f\|_\infty \|\Delta^2 f\|_\infty}, \tag{2}$$

in $\mathbb{R}^d$, $d \geq 2$, where Δf is the Laplacian and $\Delta^2 = \Delta(\Delta)$.

This result is an analogue of the famous Landau inequality ([4]) in $\mathbb{R}$:

$$\|f'\|_\infty \leq \sqrt{2}\sqrt{\|f\|_\infty \|f''\|_\infty}, \tag{3}$$

where $\|\cdot\|_\infty$ means the L_∞ norm on $\mathbb{R}^d$ (respectively on $\mathbb{R}$).

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In this article we first prove a multivariate right side Caputo fractional Taylor's formula and based on it we prove three multivariate right side Caputo fractional Landau's type inequalities where all the constants are precisely calculated and we obtain best upper bounds.

2. Background

Here we follow [1], Chapter 23.

We need

Definition 2.1. Let $f \in L_1([a,b])$, $\alpha > 0$. We define the right Riemann-Liouville fractional operator of order α by

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s)\, ds, \tag{4}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function.

We set $I_{b-}^{0} := I$ (the identity operator).

By [1], p. 334 we get that $I_{b-}^{\alpha} f(x)$ exists almost everywhere on $[a,b]$ and $I_{b-}^{\alpha} f \in L_1([a,b])$. Also if $\alpha \geq 1$ then $I_{b-}^{\alpha} f \in C([a,b])$.

We need

Definition 2.2. Let $f \in AC^m([a,b])$ (space of functions from $[a,b]$ into $\mathbb{R}$ with $(m-1)$-derivative absolutely continuous function on $[a,b]$), $m \in \mathbb{N}$, where $m = \lceil \alpha \rceil$, $\alpha > 0$ ($\lceil \cdot \rceil$ the ceiling of the number).

We define the right Caputo fractional derivative of order $\alpha > 0$, by

$$D_{b-}^{\alpha} f(x) := (-1)^m I_{b-}^{m-\alpha} f^{(m)}(x), \tag{5}$$

that is

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (s-x)^{m-\alpha-1} f^{(m)}(s)\, ds. \tag{6}$$

By [1], p. 337 we get that $D_{b-}^{\alpha} f(x)$ exists a.e. on $[a,b]$ and $D_{b-}^{\alpha} f \in L_1([a,b])$.

When $\alpha = m \in \mathbb{N}$, then

$$D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x), \tag{7}$$

$\forall\, x \in [a,b]$.

We also need the right Caputo fractional Taylor's formula with integral remainder.

Theorem 2.1. *([1], p. 341) Let $f \in AC^m([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m = \lceil \alpha \rceil$. Then*

$$f(x) = \sum_{\lambda=0}^{m-1} \frac{f^{(\lambda)}(b)}{\lambda!} (x-b)^{\lambda} + \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} D_{b-}^{\alpha} f(s)\, ds. \tag{8}$$

3. Main Results

We make

Remark. *Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \geq 2$; $z := (z_1, ..., z_k)$, $x_0 := (x_{01}, ..., x_{0k}) \in Q$. Let $f \in C^n(Q)$, $n \in \mathbb{N}$.*

Set

$$g_z(t) := f(x_0 + t(z - x_0)), \quad 0 \leq t \leq 1; \tag{9}$$
$$g_z(0) = f(x_0), \quad g_z(1) = f(z).$$

Then

$$g_z^{(j)}(t)=\left[\left(\sum_{i=1}^{k}(z_i-x_{0i})\frac{\partial}{\partial x_i}\right)^j f\right](x_0+t(z-x_0)), \tag{10}$$

and

$$g_z^{(j)}(1)=\left[\left(\sum_{i=1}^{k}(z_i-x_{0i})\frac{\partial}{\partial x_i}\right)^j f\right](z), \tag{11}$$

for $j=0,1,...,n$.

If all $f_\alpha(z):=\frac{\partial^\alpha f}{\partial x^\alpha}(z)=0$, $\alpha:=(\alpha_1,...,\alpha_k)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,k$; $|\alpha|:=\sum_{i=1}^{k}\alpha_i=l$, *then* $g_z^{(l)}(1)=0$, *where* $l\in\{0,1,...,n\}$.

Clearly here $g_z\in C^n([0,1])$, *in particular* $g_z\in AC^n([0,1])$.

We also have that

$$g_z^{(j)}(t)=\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_k),\alpha_i\in\mathbb{Z}^+\\ i=1,...,k;|\alpha|:=\sum_{i=1}^{k}\alpha_i=j}}\left(\frac{j!}{\prod_{i=1}^{k}\alpha_i!}\right)\left(\prod_{i=1}^{k}(z_i-x_{0i})^{\alpha_i}\right)f_\alpha(x_0+t(z-x_0)), \tag{12}$$

$0\le t\le 1$, *and*

$$\frac{g_z^{(j)}(1)}{j!}=\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_k),\alpha_i\in\mathbb{Z}^+\\ i=1,...,k;|\alpha|:=\sum_{i=1}^{k}\alpha_i=j}}\left(\frac{\prod_{i=1}^{k}(z_i-x_{0i})^{\alpha_i}}{\prod_{i=1}^{k}\alpha_i!}\right)f_\alpha(z), \tag{13}$$

for all $j=0,1,...,n$.

Let now $\nu>0$ *with* $\lceil\nu\rceil=n$.

By applying (8) to g_z *we find*

$$g_z(0)=\sum_{j=0}^{n-1}(-1)^j\frac{g_z^{(j)}(1)}{j!}+\frac{1}{\Gamma(\nu)}\int_0^1 s^{\nu-1}D_{1-}^{\nu}g_z(s)\,ds. \tag{14}$$

Still we need to interpret $D_{1-}^{\nu}g_z$ *as follows* ($0\le s\le 1$):

$$D_{1-}^{\nu}g_z(s)\overset{(6)}{=}\frac{(-1)^n}{\Gamma(n-\nu)}\int_s^1(y-s)^{n-\nu-1}g_z^{(n)}(y)\,dy= \tag{15}$$

$$\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_k),\alpha_i\in\mathbb{Z}^+\\ i=1,...,k;|\alpha|:=\sum_{i=1}^{k}\alpha_i=n}}\left(\frac{n!}{\prod_{i=1}^{k}\alpha_i!}\right)\left(\prod_{i=1}^{k}(z_i-x_{0i})^{\alpha_i}\right)$$

$$\frac{1}{\Gamma(n-\nu)}\int_s^1(y-s)^{n-\nu-1}f_\alpha(x_0+y(z-x_0))\,dy.$$

Thus it holds ($0 \leq s \leq 1$)

$$D_{1-}^{\nu} g_z(s) = (-1)^n \sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_k),\alpha_i\in\mathbb{Z}^+ \\ i=1,\dots,k;|\alpha|:=\sum_{i=1}^{k}\alpha_i=n}} \left(\frac{n!}{\prod_{i=1}^{k}\alpha_i!} \right) \left(\prod_{i=1}^{k} (z_i - x_{0i})^{\alpha_i} \right) \left(I_{1-}^{n-\nu} f_\alpha (x_0 + t(z - x_0)) \right)(s). \tag{16}$$

Based on the above we have proved the following multivariate right side Caputo fractional Taylor's formula:

Theorem 3.1. *Let* $\nu > 0$, $n = \lceil \nu \rceil$, $f \in C^n(Q)$, *where* Q *is a compact and convex subset of* $\mathbb{R}^k$, $k \geq 2$; *with* $x_0 := (x_{01}, \dots, x_{0k})$, $z := (z_1, \dots, z_k) \in Q$. *Then*

1)

$$f(x_0) = \sum_{j=0}^{n-1} (-1)^j \left(\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_k),\alpha_i\in\mathbb{Z}^+ \\ i=1,\dots,k;|\alpha|:=\sum_{i=1}^{k}\alpha_i=j}} \left(\frac{\prod_{i=1}^{k}(z_i - x_{0i})^{\alpha_i}}{\prod_{i=1}^{k}\alpha_i!} \right) f_\alpha(z) \right) +$$

$$(-1)^n \sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_k),\alpha_i\in\mathbb{Z}^+ \\ i=1,\dots,k;|\alpha|:=\sum_{i=1}^{k}\alpha_i=n}} \left(\frac{n!}{\prod_{i=1}^{k}\alpha_i!} \right) \left(\prod_{i=1}^{k} (z_i - x_{0i})^{\alpha_i} \right) \tag{17}$$

$$\frac{1}{\Gamma(\nu)} \int_0^1 s^{\nu-1} \left(I_{1-}^{n-\nu} f_\alpha (x_0 + t(z - x_0)) \right)(s)\, ds.$$

2) Additionally assume that $f_\alpha(z) = 0$, $\alpha := (\alpha_1, \dots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, k$, $|\alpha| := \sum_{i=1}^{k} \alpha_i =: r$, $r = 0, 1, \dots, n-1$, *then*

$$f(x_0) = (-1)^n \sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_k),\alpha_i\in\mathbb{Z}^+ \\ i=1,\dots,k;|\alpha|:=\sum_{i=1}^{k}\alpha_i=n}} \left(\frac{n!}{\prod_{i=1}^{k}\alpha_i!} \right) \left(\prod_{i=1}^{k} (z_i - x_{0i})^{\alpha_i} \right) \frac{1}{\Gamma(\nu)} \int_0^1 s^{\nu-1} \left(I_{1-}^{n-\nu} f_\alpha (x_0 + t(z - x_0)) \right)(s)\, ds. \tag{18}$$

Next we give three multivariate right side Caputo fractional Landaus' type inequalities.

Theorem 3.2. *Let $f \in C^4\left(\mathbb{R}_-^k\right)$, $k \geq 2$, with $\|f\|_{\infty,\mathbb{R}_-^k} < \infty$, and $3 < \nu < 4$. Assume further that*

$$\Phi_\nu := \sup_{x_0, z \in \mathbb{R}_-^k, s \in [0,1]} \left| \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_k), \alpha_i \in \mathbb{Z}^+ \\ i = 1, \ldots, k; |\alpha| := \sum_{i=1}^k \alpha_i = 4}} \frac{I_{1-}^{4-\nu}\left(f_\alpha\left(x_0 + t\left(z - x_0\right)\right)\right)(s)}{\prod_{i=1}^k \alpha_i!} \right| < +\infty. \tag{19}$$

Then we derive the best upper bounds:

1)
$$\left\| \sum_{i=1}^k \frac{\partial f}{\partial x_i} \right\|_{\infty,\mathbb{R}_-^k} \leq 4 \left(\frac{11}{9}\right)^{\frac{3}{4}} \left(\frac{144}{\Gamma(\nu+1)}\right)^{\frac{1}{4}} \|f\|_{\infty,\mathbb{R}_-^k}^{\frac{3}{4}} \Phi_\nu^{\frac{1}{4}} < +\infty, \tag{20}$$

2)
$$\left\| \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_k), \alpha_i \in \mathbb{Z}^+ \\ i = 1, \ldots, k; |\alpha| := \sum_{i=1}^k \alpha_i = 2}} \frac{f_\alpha}{\prod_{i=1}^k \alpha_i!} \right\|_{\infty,\mathbb{R}_-^k} \leq 2\sqrt{\frac{528}{\Gamma(\nu+1)}} \sqrt{\|f\|_{\infty,\mathbb{R}_-^k} \Phi_\nu} < +\infty, \tag{21}$$

and

3)
$$\left\| \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_k), \alpha_i \in \mathbb{Z}^+ \\ i = 1, \ldots, k; |\alpha| := \sum_{i=1}^k \alpha_i = 3}} \frac{f_\alpha}{\prod_{i=1}^k \alpha_i!} \right\|_{\infty,\mathbb{R}_-^k} \leq \frac{4}{3} \left(\frac{144}{\Gamma(\nu+1)}\right)^{\frac{3}{4}} \|f\|_{\infty,\mathbb{R}_-^k}^{\frac{1}{4}} \Phi_\nu^{\frac{3}{4}} < +\infty. \tag{22}$$

Proof. We set

$$\Delta_1(z) := \sum_{i=1}^k \frac{\partial f(z)}{\partial x_i},$$
$$\Delta_2(z) := \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_k), \alpha_i \in \mathbb{Z}^+ \\ i = 1, \ldots, k; |\alpha| := \sum_{i=1}^k \alpha_i = 2}} \frac{f_\alpha(z)}{\prod_{i=1}^k \alpha_i!}, \tag{23}$$

and

$$\Delta_3(z) := \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_k), \alpha_i \in \mathbb{Z}^+ \\ i = 1, \ldots, k; |\alpha| := \sum_{i=1}^k \alpha_i = 3}} \frac{f_\alpha(z)}{\prod_{i=1}^k \alpha_i!}.$$

We can choose $x_0, z \in \mathbb{R}_-^k$ so that $z_i - x_{0i} = -h$, all $i = 1, \ldots, k$, where $h > 0$. By Theorem 3.1 we get

$$f(x_0) = f(z) + \Delta_1(z) h + \Delta_2(z) h^2 + \Delta_3(z) h^3 + R_4 h^4, \tag{24}$$

where

$$R_4 := \frac{1}{\Gamma(\nu)} \left[\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_k),\alpha_i\in\mathbb{Z}^+ \\ i=1,\ldots,k;|\alpha|:=\sum_{i=1}^{k}\alpha_i=4}} \left(\frac{4!}{\prod_{i=1}^{k}\alpha_i!} \right) \right.$$
$$\left. \int_0^1 s^{\nu-1} I_{1-}^{4-\nu} \left(f_\alpha \left(x_0 + t \left(z - x_0 \right) \right) \right) (s)\, ds \right]. \tag{25}$$

Therefore, $\forall\ h>0$, we get

$$\begin{array}{l} \Delta_1(z)\,h + \Delta_2(z)\,h^2 + \Delta_3(z)\,h^3 = f(x_0) - f(z) - R_4 h^4 =: A_1, \\ \Delta_1(z)\,2h + \Delta_2(z)\,4h^2 + \Delta_3(z)\,8h^3 = f(x_0) - f(z) - R_4 16 h^4 =: A_2, \\ \text{and} \\ \Delta_1(z)\,3h + \Delta_2(z)\,9h^2 + \Delta_3(z)\,27h^3 = f(x_0) - f(z) - R_4 81 h^4 =: A_3. \end{array} \tag{26}$$

We solve the system of three equations (26) and we find:

$$\begin{array}{l} \Delta_1(z) = \frac{18A_1 - 9A_2 + 2A_3}{6h}, \\ \Delta_2(z) = \frac{4A_2 - 5A_1 - A_3}{2h^2}, \\ \text{and} \\ \Delta_3(z) = \frac{3A_1 - 3A_2 + A_3}{6h^3}. \end{array} \tag{27}$$

We have also that

$$|R_4| \le \frac{24\Phi_\nu}{\Gamma(\nu+1)}. \tag{28}$$

We calculate

$$18A_1 - 9A_2 + 2A_3 = 11\left(f(x_0) - f(z)\right) - 36R_4h^4. \tag{29}$$

Therefore it holds

$$|\Delta_1(z)| \le \frac{11\,\|f\|_{\infty,\mathbb{R}_-^k}}{3h} + \frac{144\Phi_\nu}{\Gamma(\nu+1)}h^3, \tag{30}$$

$\forall\ h>0, \forall\ z\in\mathbb{R}_-^k$.

We get that

$$\|\Delta_1\|_{\infty,\mathbb{R}_-^k} \le \frac{11\,\|f\|_{\infty,\mathbb{R}_-^k}}{3h} + \left(\frac{144\Phi_\nu}{\Gamma(\nu+1)}\right)h^3, \tag{31}$$

$\forall\ h>0$.

Next we find

$$4A_2 - 5A_1 - A_3 = -2\left(f(x_0) - f(z)\right) + 22R_4h^4. \tag{32}$$

Hence

$$|\Delta_2(z)| \le \frac{2\,\|f\|_{\infty,\mathbb{R}_-^k}}{h^2} + \left(\frac{264\Phi_\nu}{\Gamma(\nu+1)}\right)h^2, \tag{33}$$

$\forall\ h>0, \forall\ z\in\mathbb{R}_-^k$.

The last implies

$$\|\Delta_2\|_{\infty,\mathbb{R}_-^k} \le \frac{2\,\|f\|_{\infty,\mathbb{R}_-^k}}{h^2} + \left(\frac{264\Phi_\nu}{\Gamma(\nu+1)}\right)h^2, \tag{34}$$

$\forall\ h>0$.

We also calculate

$$3A_1 - 3A_2 + A_3 = (f(x_0) - f(z)) - 36R_4h^4.$$

Therefore

$$|\Delta_3(z)| \leq \frac{\|f\|_{\infty,\mathbb{R}_-^k}}{3h^3} + \left(\frac{144\Phi_\nu}{\Gamma(\nu+1)}\right)h, \tag{35}$$

$\forall\, h > 0, \forall\, z \in \mathbb{R}_-^k$.

The last implies

$$\|\Delta_3\|_{\infty,\mathbb{R}_-^k} \leq \frac{\|f\|_{\infty,\mathbb{R}_-^k}}{3h^3} + \left(\frac{144\Phi_\nu}{\Gamma(\nu+1)}\right)h, \tag{36}$$

$\forall\, h > 0$.

We will work on (31).

Call $\mu := \frac{11\|f\|_{\infty,\mathbb{R}_-^k}}{3}$, $\theta := \left(\frac{144\Phi_\nu}{\Gamma(\nu+1)}\right)$, both are positive.

We study

$$y(h) = \frac{\mu}{h} + \theta h^3, \ \ \forall\, h > 0. \tag{37}$$

We have

$$y'(h) = -\mu h^{-2} + 3\theta h^2 = 0, \tag{38}$$

with one critical number

$$h_{crit.no.} = h_0 = \left(\frac{\mu}{3\theta}\right)^{\frac{1}{4}} > 0. \tag{39}$$

We have that

$$y''(h_0) = 2\mu h_0^{-3} + 6\theta h_0 > 0, \tag{40}$$

implying that y has a global minimum. We see that

$$y(h_0) = \mu h_0^{-1} + \theta h_0^3 = h_0^3\left(\mu h_0^{-4} + \theta\right) = \frac{4}{3^{\frac{3}{4}}}\theta^{\frac{1}{4}}\mu^{\frac{3}{4}}. \tag{41}$$

I.e.

$$y(h_0) = \frac{4}{3^{\frac{3}{4}}}\theta^{\frac{1}{4}}\mu^{\frac{3}{4}}, \tag{42}$$

is the global minimum.

Consequently

$$\|\Delta_1\|_{\infty,\mathbb{R}_-^k} \leq 4\left(\frac{11}{9}\right)^{\frac{3}{4}}\left(\frac{144}{\Gamma(\nu+1)}\right)^{\frac{1}{4}}\|f\|_{\infty,\mathbb{R}_-^k}^{\frac{3}{4}}\Phi_\nu^{\frac{1}{4}}. \tag{43}$$

Next we work on (34).

Call $\varphi := 2\|f\|_{\infty,\mathbb{R}_-^k}$, $\psi := \frac{264\Phi_\nu}{\Gamma(\nu+1)} > 0$.

We study

$$y(h) := \frac{\varphi}{h^2} + \psi h^2, \ \ \forall\, h > 0. \tag{44}$$

We have

$$y'(h) = -2\varphi h^{-3} + 2\psi h = 0, \tag{45}$$

with

$$h_{crit.no.} = h_0 = \left(\frac{\varphi}{\psi}\right)^{\frac{1}{4}} > 0. \tag{46}$$

Notice that

$$y''(h_0) = 6\varphi h_0^{-4} + 2\psi > 0. \tag{47}$$

Thus we have that y has global minimum which is

$$y(h_0) = \varphi h_0^{-2} + \psi h_0^2 = h_0^2 \left(\varphi h_0^{-4} + \psi\right) = 2\psi^{\frac{1}{2}} \varphi^{\frac{1}{2}},$$

that is

$$y(h_0) = 2\psi^{\frac{1}{2}} \varphi^{\frac{1}{2}}. \tag{48}$$

Therefore we derive

$$\|\Delta_2\|_{\infty,\mathbb{R}_-^k} \leq 2\sqrt{\frac{528}{\Gamma(\nu+1)}} \sqrt{\|f\|_{\infty,\mathbb{R}_-^k} \Phi_\nu}. \tag{49}$$

Finally we work on (36).

Let $A := \frac{\|f\|_{\infty,\mathbb{R}_-^k}}{3}$, $B := \frac{144\Phi_\nu}{\Gamma(\nu+1)}$, both are positive.

We study

$$y(h) := \frac{A}{h^3} + Bh, \ \forall\, h > 0. \tag{50}$$

Then

$$y'(h) = -3Ah^{-4} + B = 0, \tag{51}$$

so that

$$h_{crit.no.} = h_0 = \left(\frac{3A}{B}\right)^{\frac{1}{4}} > 0. \tag{52}$$

Here

$$y''(h_0) = 12Ah_0^{-5} > 0. \tag{53}$$

Hence y has global minimum which is

$$y(h_0) = Ah_0^{-3} + Bh_0 = h_0\left(Ah_0^{-4} + B\right) = \frac{4}{3^{\frac{3}{4}}} A^{\frac{1}{4}} B^{\frac{3}{4}}.$$

That is

$$y(h_0) = \frac{4}{3^{\frac{3}{4}}} A^{\frac{1}{4}} B^{\frac{3}{4}}, \tag{54}$$

the global minimum of y.

Consequently it holds

$$\|\Delta_3\|_{\infty,\mathbb{R}_-^k} \leq \frac{4}{3}\left(\frac{144}{\Gamma(\nu+1)}\right)^{\frac{3}{4}} \|f\|_{\infty,\mathbb{R}_-^k}^{\frac{1}{4}} \Phi_\nu^{\frac{3}{4}}. \tag{55}$$

The theorem is proved. □

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