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NEW OPERATORS FOR FERMATEAN FUZZY SETS

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ABSTRACT. In this paper, we define some new operators $[(A@B),(A\$B),(A\#B),(A*B),(A\rightarrow B)]$ of Fermatean fuzzy sets. Then we discuss several properties of these operators. Further we prove necessity and possibility operators of Fermatean fuzzy sets and investigates the algebraic properties. Finally, we have identified and proved several of these properties, particularly those involving the operator $A\rightarrow B$ defined as Fermatean fuzzy implication with other operators.

1. INTRODUCTION AND PRELIMINARIES

Attanasov [1] proposed the intuitionistic fuzzy set (IFS) $A = x, \mu_A(x), \nu_A(x)|x \in X$, where $\mu_A(x) \in [0,1]$ represent the membership degree and $\nu_A(x) \in [0,1]$ the non-membership degree for all $x \in X$, respectively. Since the IFS was proposed, it has received a lot of attention in many fields, such as pattern recognition, medical diagnosis, and so on. But if the sum of the membership degree and the nonmembership degree is greater than 1, the IFS is no longer applicable. Yager [8] proposed the Pythagorean fuzzy set (PFS) $A = x, \mu_A(x), \nu_A(x)|x \in X$, where the squared sum of its membership degree $\mu_A(x) \in [0,1]$ and nonmembership degree $\nu_A(x) \in [0,1]$ is less than or equal to 1. Since the PFS was brought up, it has been widely applied in different fields, such as investment decision making, service quality of domestic airline, collaborative-based recommender systems, and so on. Although the PFS generalizes the IFS, it cannot describe the following decision information. A panel of experts were invited to give their opinions about the feasibility of an investment plan, and they were divided into two independent groups to make a decision. One group considered the degree of the feasibility of the investment plan as 0.9, while the other group considered the nonmembership degree as 0.6. It was clearly seen that $0.9+0.6 > 1, (0.9)^2+(0.6)^2 > 1$, and thus it could not be described by IFS and PFS. After the IFS and PFS theory, many researchers [2, 3, 5, 7, 9] attempted the important role in this theory. To describe such evaluation information, Senapati and Yager [4] proposed Fermatean fuzzy set (FFS) $A = x, \mu_A(x), \nu_A(x)|x \in X$, where represent the $\mu_A(x) \in [0,1]$ membership degree and $\nu_A(x) \in [0,1]$ the non-membership degree for all $x \in X$, respectively, and $0 \leq \mu_A^3(x) + \nu_A^3(x) \leq 1$. It was clearly seen that

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$0.9+0.6>1, (0.9)^2+(0.6)^2>1, (0.9)^3+(0.6)^3\leq 1$. In this paper we have developed some new operators for Fermatean fuzzy sets and discussed several properties.

Definition 1.1. [4] A Fermatean fuzzy set A on a universe X is an object of the form $A=\{(x,\mu_A(x),\nu_A(x))\,|x\in X\}$, where $\mu_A(x)\in[0,1]$ is called the degree of membership of x in A, $\nu_A(x)\in[0,1]$ is called the degree of non-membership of x in A, and where $\mu_A(x)$ and $\nu_A(x)$ satisfy the following condition: $0\leq\mu_A^3(x)+\nu_A^3(x)\leq 1$ for all $x\in X$

Definition 1.2. [4]. **IFS operations on FFS**.

Let $FFS(X)$ denote the family of all $FFSs$ on the universe X, and let $A,B\in FFS(X)$ be given as

$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle\,|x\in X\}$,
$B=\{\langle x,\mu_B(x),\nu_B(x)\rangle\,|x\in X\}$.

Then following FFS operations are defined,

(i) $A\cup B=\{\langle x,\max(\mu_A(x),\mu_B(x)),\min(\nu_A(x),\nu_B(x))\rangle\,|x\in X\}$
(ii) $A\cap B=\{\langle x,\min(\mu_A(x),\mu_B(x)),\max(\nu_A(x),\nu_B(x))\rangle\,|x\in X\}$
(iii) $A^C=\{\langle x,(\nu_A(x)),(\mu_A(x))\rangle\,|x\in X\}$
(iv) $A\boxplus_h B=\left\{\left(x,\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)},\nu_A(x)\nu_B(x)\right)|x\in H\right\}$
(v) $A\boxtimes_h B=\left\{\left(x,\mu_A(x)\mu_B(x),\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right)|x\in H\right\}$.

Lemma 1.1. [2]. *For any two numbers $a,b\in[0,1]$,then*

$$a.b\leq\min\{a,b\}\leq\frac{2(a.b)}{a+b}\leq\sqrt{a.b}\leq\max\{a,b\}\leq a+b-a.b,$$
$$a.b\leq\frac{a+b}{2(a+b+1)}\leq\frac{a+b}{2}.$$

2. New Fermatean fuzzy operators

In this section, we define the new Fermatean fuzzy operators and investigates the algebraic properties.

Definition 2.1. IFS operations on FFS

Let $FFS(X)$ denote the family of all $FFSs$ on the universe X, and let $A,B\in FFS(X)$ be given as $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle\,|x\in X\}$, $B=\{\langle x,\mu_B(x),\nu_B(x)\rangle\,|x\in X\}$.

Then following FFS operations are defined,

(i) $A@B=\left\{\left\langle x,\sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}},\sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}\right\rangle|x\in X\right\}$
(ii) $A\$B=\left\{\left\langle x,\sqrt[3]{\mu_A(x)\mu_B(x)},\sqrt[3]{\nu_A(x)\nu_B(x)}\right\rangle|x\in X\right\}$
(iii) $A\#B=\left\{\left\langle x,\frac{\sqrt[3]{2}\mu_A(x)\mu_B(x)}{\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)}},\frac{\sqrt[3]{2}\nu_A(x)\nu_B(x)}{\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)}}\right\rangle|x\in X\right\}$

For which we shall accept that if $\mu_A(x)=\mu_B(x)=0$ then $\frac{\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)}=0$ and if $\nu_A(x)=\nu_B(x)=0$, then $\frac{\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)}=0$.

(iv) $A*B=\left\{\left\langle x,\sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2(\mu_A^3(x)+\mu_B^3(x)+1)}},\sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2(\nu_A^3(x)+\nu_B^3(x)+1)}}\right\rangle|x\in X\right\}$
(v) $A\rightarrow B=\{\langle x,\max(\nu_A(x),\mu_B(x)),\min(\mu_A(x),\nu_B(x))\rangle\,|x\in X\}$.

Remark. *Clearly, for each two FFSs A and B,* $[(A@B),(A\$B),(A\#B),(A*B),(A\rightarrow B)]$ *are as yet an FFS. Some basic representations are appear as follows:*
For (i),

$$0\leq\left(\sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}\right)^3+\left(\sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}\right)^3$$
$$=\frac{\mu_A^3(x)+\nu_A^3(x)}{2}+\frac{\mu_B^3(x)+\nu_B^3(x)}{2}\leq\frac{1}{2}+\frac{1}{2}=1.$$

For (ii),

If $\nu_A(x)\geq\mu_B(x)$ *and* $\mu_A(x)\geq\nu_B(x)$*, then*
$$0\leq\max\left\{\nu_A^3(x),\mu_B^3(x)\right\}+\min\left\{\mu_A^3(x),\nu_B^3(x)\right\}$$
$$\leq\nu_A^3(x)+\mu_A^3(x)\leq 1.$$

If $\nu_A(x)\geq\mu_B(x)$ *and* $\mu_A(x)\leq\nu_B(x)$*, then*
$$0\leq\max\left\{\nu_A^3(x),\mu_B^3(x)\right\}+\min\left\{\mu_A^3(x),\nu_B^3(x)\right\}$$
$$\leq\nu_A^3(x)+\mu_A^3(x)\leq 1.$$

If $\nu_A(x)\leq\mu_B(x)$ *and* $\mu_A(x)\geq\nu_B(x)$*, then*
$$0\leq\max\left\{\nu_A^3(x),\mu_B^3(x)\right\}+\min\left\{\mu_A^3(x),\nu_B^3(x)\right\}$$
$$\leq\nu_B^3(x)+\mu_B^3(x)\leq 1.$$

If $\nu_A(x)\leq\mu_B(x)$ *and* $\mu_A(x)\leq\nu_B(x)$*, then*
$$0\leq\max\left\{\nu_A^3(x),\mu_B^3(x)\right\}+\min\left\{\mu_A^3(x),\nu_B^3(x)\right\}$$
$$\leq\nu_B^3(x)+\mu_B^3(x)\leq 1.$$

For (iii),

$$0\leq\left(\sqrt[3]{\mu_A(x)\mu_B(x)}\right)^3+\left(\sqrt[3]{\nu_A(x)\nu_B(x)}\right)^3=\mu_A(x)\mu_B(x)+\nu_A(x)\nu_B(x)$$
$$=\frac{\mu_A^3(x)+\nu_A^3(x)}{2}+\frac{\mu_B^3(x)+\nu_B^3(x)}{2}$$
$$\leq\frac{1}{2}+\frac{1}{2}=1.$$

For (iv),

$$0\leq\left(\frac{\sqrt[3]{2}\mu_A(x)\mu_B(x)}{\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)}}\right)^3+\left(\frac{\sqrt[3]{2}\nu_A(x)\nu_B(x)}{\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)}}\right)^3$$
$$=\frac{2\mu_A^3(x)\mu_B^3(x)}{\mu_A^3(x)+\mu_B^3(x)}+\frac{2\nu_A^3(x)\nu_B^3(x)}{\nu_A^3(x)+\nu_B^3(x)}\leq 1.$$

For (v),

$$0\leq\left(\sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2(\mu_A^3(x)+\mu_B^3(x)+1)}}\right)^3+\left(\sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2(\nu_A^3(x)+\nu_B^3(x)+1)}}\right)^3$$
$$=\frac{\mu_A^3(x)+\mu_B^3(x)}{2(\mu_A^3(x)+\mu_B^3(x)+1)},\frac{\nu_A^3(x)+\nu_B^3(x)}{2(\nu_A^3(x)+\nu_B^3(x)+1)}\leq 1.$$

Theorem 2.1. *For* $A,B\in FFS(X)$*,*
$(i)A@B=B@A=(A^C@B^C)^C$
$(ii)A\$B=B\$A=(A^C\$B^C)^C$
$(iii)A\#B=B\#A=(A^C\#B^C)^C$
$(iv)A*B=B*A=(A^C*B^C)^C$

Proof. Let (i) is prove, then other can be proved similarly.
(i) Let A and B be two given FFSs, then

$$A@B = \left\{ \left\langle x, \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}} \right\rangle |x \in X \right\}$$
$$= \left\{ \left\langle x, \sqrt[3]{\frac{\mu_B^3(x)+\mu_A^3(x)}{2}}, \sqrt[3]{\frac{\nu_B^3(x)+\nu_A^3(x)}{2}} \right\rangle |x \in X \right\}$$
$$= B@A.$$
$$A^C@B^C = \left\{ \left\langle x, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}, \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}} \right\rangle |x \in X \right\}$$
$$(A^C@B^C)^C = \left\{ \left\langle x, \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}} \right\rangle |x \in X \right\}$$
$$= A@B.$$

Hence, $A@B = B@A = (A^C@B^C)^C$. □

The following theorems are obvious.

Theorem 2.2. *For* $A, B, C \in FFS(X)$,
(i) $(A \cap B)@C = (A@C) \cap (B@C)$;
(ii) $(A \cup B)@C = (A@C) \cup (B@C)$;
(iii) $(A \cap B)\$C = (A\$C) \cap (B\$C)$;
(iv) $(A \cup B)\$C = (A\$C) \cup (B\$C)$;
(v) $(A \cap B)\#C = (A\#C) \cap (B\#C)$;
(vi) $(A \cup B)\#C = (A\#C) \cup (B\#C)$;
(vii) $(A \cap B) * C = (A * C) \cap (B * C)$;
$(viii)$ $(A \cup B) * C = (A * C) \cup (B * C)$;

Theorem 2.3. *For* $A, B, C \in FFS(X)$,
(i) $(A \boxplus_F B)@C \subseteq (A@C) \boxplus_F (B@C)$;
(ii) $(A \boxtimes_F B)@C \supseteq (A@C) \boxtimes_F (B@C)$;
(iii) $(A \boxplus_F B)\$C \subseteq (A\$C) \boxplus_F (B\$C)$;
(vi) $(A \boxtimes_F B)\$C \supseteq (A\$C) \boxtimes_F (B\$C)$;
(v) $(A \boxplus_F B) * C \subseteq (A * C) \boxplus_F (B * C)$;
(vi) $(A \boxtimes_F B) * C \supseteq (A * C) \boxtimes_F (B * C)$;

Theorem 2.4. *For* $A, B, C \in FFS(X)$,
(i) $(A@B) \boxplus_F C = (A \boxplus_F C)@(B \boxplus_F C)$;
(ii) $(A@B) \boxtimes_F C = (A \boxtimes_F C)@(B \boxtimes_F C)$;
(iii) $(A\$B) \boxplus_F C \subseteq (A \boxplus_F C)\$(B \boxplus_F C)$;
(iv) $(A\$B) \boxtimes_F C \supseteq (A \boxtimes_F C)\$(B \boxtimes_F C)$;
(v) $(A\#B) \boxplus_F C \subseteq (A \boxplus_F C)\#(B \boxplus_F C)$;
(vi) $(A\#B) \boxtimes_F C \supseteq (A \boxtimes_F C)\#(B \boxtimes_F C)$;
(vii) $(A * B) \boxplus_F C \subseteq (A \boxplus_F C) * (B \boxplus_F C)$;
$(viii)$ $(A * B) \boxtimes_F C \supseteq (A \boxtimes_F C) * (B \boxtimes_F C)$;

3. NECESSITY AND POSSIBILITY OPERATORS ON FERMATEAN FUZZY SETS

In this section, we prove the necessity and possibility operators of Fermatean fuzzy sets. Then we compile some relevent properties of these operators are discussed.

Definition 3.1. [6] The necessity and possibility operators on a Fermatean fuzzy set A is denoted by $\Box A, \Diamond A$ and is
$(i)\ \Box A = \Big\{x, \Big\langle \mu_A(x),\ \sqrt[3]{1-\mu_A^3(x)} \Big\rangle\, |x \in X\Big\}$,
$(ii)\ \Diamond A = \Big\{x, \Big\langle \sqrt[3]{1-\nu_A^3(x)}, \nu_A(x) \Big\rangle\, |x \in X\Big\}$.

Theorem 3.1. *For $A, B \in FFS(X)$,*
$(i)\ \Box A @ \Box B = \Box(A@B) \subseteq \Diamond A @ \Diamond B = \Diamond(A@B)$;
$(ii)\ \Box(A\$B) \subseteq \Box A \$ \Box B \subseteq \Diamond A \$ \Diamond B \subseteq \Diamond(A\$B)$;
$(iii)\ \Box(A\#B) \subseteq \Box A \# \Box B \subseteq \Diamond A \# \Diamond B \subseteq \Diamond(A\#B)$;
$(iv)\ \Box(A * B) \subseteq \Box A * \Box B \subseteq \Diamond A * \Diamond B \subseteq \Diamond(A * B)$.

Proof. Let (i) and (iii) are proved, then other can be proved similarly.
(i) Let $\Box A @ \Box B$

$$= \left\{\left\langle x, \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \sqrt[3]{\frac{1-\mu_A^3(x)+1-\mu_B^3(x)}{2}} \right\rangle |x \in X\right\}$$

$= \Box(A@B)$.

$\Diamond A @ \Diamond B$

$$= \left\{\left\langle x, \sqrt[3]{\frac{1-\nu_A^3(x)+1-\nu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}} \right\rangle |x \in X\right\}$$

$= \Diamond(A@B)$.

Hence, $\Box A @ \Box B = \Box(A@B) \subseteq \Diamond A @ \Diamond B = \Diamond(A@B)$.

(iii) Let $\Box A \# \Box B$

$$= \left\{\left\langle x, \frac{\sqrt[3]{2}\mu_A(x)\mu_B(x)}{\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)}}, \frac{\sqrt[3]{2}\,\sqrt[3]{1-\mu_A^3(x)}\,\sqrt[3]{1-\mu_B^3(x)}}{\sqrt[3]{1-\mu_A^3(x)+1-\mu_B^3(x)}} \right\rangle |x \in X\right\}$$

$\subseteq \Box(A\#B)$.

$\Diamond A \# \Diamond B$

$$= \left\{\left\langle x, \frac{\sqrt[3]{2}\,\sqrt[3]{1-\nu_A^3(x)}\,\sqrt[3]{1-\nu_B^3(x)}}{\sqrt[3]{1-\nu_A^3(x)+1-\nu_B^3(x)}}, \frac{\sqrt[3]{2}\nu_A(x)\nu_B(x)}{\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)}} \right\rangle |x \in X\right\}$$

$\subseteq \Diamond(A\#B)$.

Hence, $\Box(A\#B) \subseteq \Box A \# \Box B \subseteq \Diamond A \# \Diamond B \subseteq \Diamond(A\#B)$. $\Box$

The following theorems are obvious.

Theorem 3.2. *For $A, B \in FFS(X)$,*
$(i)\ \Box\Big[(\Diamond A @ \Diamond B)^C\Big] = \Big[\Diamond(A@B)\Big]^C$;
$(ii)\ \Diamond\Big[(\Box A @ \Box B)^C\Big] = \Big[\Box(A@B)\Big]^C$;
$(iii)\ \Box\Big[(\Diamond A \$ \Diamond B)^C\Big] = \Big[\Diamond(A\$B)\Big]^C$;
$(vi)\ \Diamond\Big[(\Box A \$ \Box B)^C\Big] = \Big[\Box(A\$B)\Big]^C$;
$(v)\ \Box\Big[(\Diamond A \# \Diamond B)^C\Big] = \Big[\Diamond(A\#B)\Big]^C$;
$(vi)\ \Diamond\Big[(\Box A \# \Box B)^C\Big] = \Big[\Box(A\#B)\Big]^C$;

(vii) $\Box\Big[(\Diamond A * \Diamond B)^C\Big] = \Big[\Diamond(A * B)\Big]^C$;
$(viii)$ $\Diamond\Big[(\Box A * \Box B)^C\Big] = \Big[\Box(A * B)\Big]^C$.

Theorem 3.3. *For* $A, B \in FFS(X)$*,*
(i) $\Big[(\Box A \boxplus_F \Diamond B)^C @((\Box A)^C \boxtimes_F \Diamond B)\Big] \cup (\Box A)^C = (\Box A)^C$;
(ii) $\Big[(\Box A \boxtimes_F \Diamond B)^C @((\Box A)^C \boxplus_F \Diamond B)\Big] \cap (\Box A)^C = (\Box A)^C$;
(iii) $\Big[(\Box A \boxplus_F \Diamond B)^C \$((\Box A)^C \boxtimes_F \Diamond B)\Big] \cup (\Box A)^C = (\Box A)^C$;
(iv) $\Big[(\Box A \boxtimes_F \Diamond B)^C \$((\Box A)^C \boxplus_F \Diamond B)\Big] \cap (\Box A)^C = (\Box A)^C$;
(v) $\Big[(\Box A \boxplus_F \Diamond B)^C \#((\Box A)^C \boxtimes_F \Diamond B)\Big] \cup (\Box A)^C = (\Box A)^C$;
(vi) $\Big[(\Box A \boxtimes_F \Diamond B)^C \#((\Box A)^C \boxplus_F \Diamond B)\Big] \cap (\Box A)^C = (\Box A)^C$;
(vii) $\Big[(\Diamond A \boxplus_F \Box B)^C @((\Box A)^C \boxtimes_F \Diamond B)\Big] \cup (\Diamond A)^C = (\Diamond A)^C$;
$(viii)$ $\Big[(\Diamond A \boxplus_F \Box B)^C \$((\Box A)^C \boxtimes_F \Diamond B)\Big] \cup (\Diamond A)^C = (\Diamond A)^C$;
(ix) $\Big[(\Diamond A \boxplus_F \Box B)^C \#((\Box A)^C \boxtimes_F \Diamond B)\Big] \cup (\Diamond A)^C = (\Diamond A)^C$;
(x) $\Big[(\Diamond A \boxplus_F \Box B)^C @((\Diamond A)^C \boxtimes_F \Box B)\Big] \cup (\Diamond A)^C = (\Diamond A)$;
(xi) $\Big[(\Diamond A \boxplus_F \Box B)^C \$((\Diamond A)^C \boxtimes_F \Box B)\Big] \cup (\Diamond A)^C = (\Diamond A)$;
(xii) $\Big[(\Diamond A \boxplus_F \Box B)^C \#((\Diamond A)^C \boxtimes_F \Box B)\Big] \cup (\Diamond A)^C = (\Diamond A)$.

In the next section, we state and prove some new results involving implication operator with other FFS operators

4. Fermatean fuzzy implication operator

In this section, the proofs of the following theorems and corollaries follows from the Definitions (1.2), (2.1) and Lemma (1.1).

Theorem 4.1. *For* $A, B \in FFS(X)$*,*
(i) $\big(A^C \to B\big) @ \big(A \to B^C\big)^C = (A@B)$,
(ii) $\big(A^C \to B\big) \boxplus_F \big(A \to B^C\big)^C = (A \boxplus_F B)$,
(iii) $\big(A^C \to B\big) \boxtimes_F \big(A \to B^C\big)^C = (A \boxtimes_F B)$,
(iv) $\big(A^C \to B\big) \$ \big(A \to B^C\big)^C = (A\$B)$,
(v) $\big(A^C \to B\big) \# \big(A \to B^C\big)^C = (A\#B)$,
(vi) $(A \to B)^C \boxplus_F (B \to A) = \big(A \boxplus_F B^C\big)$,
(vii) $(A \to B)^C @ (B \to A) = \big(A@B^C\big)$,
$(viii)$ $(A \to B)^C \boxtimes_F (B \to A) = \big(A \boxtimes_F B^C\big)$,
(ix) $(A \to B)^C \$ (B \to A) = \big(A\$B^C\big)$,
(x) $(A \to B)^C \# (B \to A) = \big(A\#B^C\big)$.

Proof. We prove (i) and (vi), results $(iii), (iv), (v), (vii), (viii), (ix), (x)$ can be proved analogously.
(i) Let $\big(A^C \to B\big) @ \big(A \to B^C\big)^C$

$$= \left[x, \sqrt[3]{\frac{\max\left\{\mu_A^3(x), \mu_B^3(x)\right\} + \min\left\{\mu_A^3(x), \mu_B^3(x)\right\}}{2}}, \right.$$
$$\left. \sqrt[3]{\frac{\min\left\{\nu_A^3(x), \nu_B^3(x)\right\} + \max\left\{\nu_A^3(x), \nu_B^3(x)\right\}}{2}} | x \in X\right]$$
$$= \left\{x, \left\langle \sqrt[3]{\frac{\mu_A^3(x) + \mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x) + \nu_B^3(x)}{2}} \right\rangle | x \in X\right\}$$
$$= (A@B).$$

(vi) Let $(A \to B)^C \boxplus_F (B \to A)$

$$= \left[x, \sqrt[3]{\min\{\mu_A^3(x), \nu_B^3(x)\} + \max\{\nu_B^3(x), \mu_A^3(x)\}} - \right.$$
$$\sqrt[3]{\min\{\mu_A^3(x), \nu_B^3(x)\} \max\{\nu_B^3(x), \mu_A^3(x)\}},$$
$$\left. \max\{\nu_A(x), \mu_B(x)\} \min\{\mu_B(x), \nu_A(x)\} | x \in X\right]$$
$$= \left\{x, \left\langle \sqrt[3]{\mu_A^3(x) + \nu_B^3(x) - \mu_A^3(x)\nu_B^3(x)}, \nu_A(x)\mu_B(x) \right\rangle | x \in X\right\}$$
$$= (A \boxplus_F B^C).$$

□

Theorem 4.2. *For $A, B \in FFS(X)$,*

(i) $\big((A \boxplus_F B) \to (A@B)^C\big)^C = \big((A@B) \to (A \boxplus_F B)^C\big)^C = (A@B)$,
(ii) $\big((A \boxplus_F B)^C \to (A@B)\big) = \big((A@B)^C \to (A \boxplus_F B)\big) = (A \boxplus_F B)$,
(iii) $\big((A \boxtimes_F B) \to (A@B)^C\big)^C = \big((A@B) \to (A \boxtimes_F B)^C\big)^C = (A \boxtimes_F B)$,
(iv) $\big((A \boxtimes_F B)^C \to (A@B)\big) = \big((A@B)^C \to (A \boxtimes_F B)\big) = (A@B)$,
(v) $\big((A \boxplus_F B) \to (A\#B)^C\big)^C = \big((A\#B) \to (A \boxplus_F B)^C\big)^C = (A\#B)$,
(vi) $\big((A \boxplus_F B)^C \to (A\#B)\big) = \big((A\#B)^C \to (A \boxplus_F B)\big) = (A \boxplus_F B)$,
(vii) $\big((A \boxtimes_F B) \to (A\#B)^C\big)^C = \big((A\#B) \to (A \boxtimes_F B)^C\big)^C = (A \boxtimes_F B)$,
$(viii)$ $\big((A \boxtimes_F B)^C \to (A\#B)\big) = \big((A\#B)^C \to (A \boxtimes_F B)\big) = (A\#B)$,
(ix) $\big((A \boxplus_F B) \to (A\$B)^C\big)^C = \big((A\$B) \to (A \boxplus_F B)^C\big)^C = (A\$B)$,
(x) $\big((A \boxplus_F B)^C \to (A\$B)\big) = \big((A\$B)^C \to (A \boxplus_F B)\big) = (A \boxplus_F B)$,
(xi) $\big((A \boxtimes_F B) \to (A\$B)^C\big)^C = \big((A\$B) \to (A \boxtimes_F B)^C\big)^C = (A \boxtimes_F B)$,
(xii) $\big((A \boxtimes_F B)^C \to (A\$B)\big) = \big((A\$B)^C \to (A \boxtimes_F B)\big) = (A\$B)$,
$(xiii)$ $\big((A \boxplus_F B) \to (A \boxtimes_F B)^C\big)^C = \big((A \boxtimes_F B) \to (A \boxplus_F B)^C\big)^C = (A \boxtimes_F B)$,
(xiv) $\big((A \boxtimes_F B)^C \to (A \boxplus_F B)\big) = \big((A \boxplus_F B)^C \to (A \boxtimes_F B)\big) = (A \boxplus_F B)$.

Proof. We prove (i), (iii), (v), (vii), (ix) and (xiii), other results can be proved analogously,

(i) Let $\big((A \boxplus_F B) \to (A@B)^C\big)^C$

$$= \left[x, \min\left\{ \sqrt[3]{\mu_A^3(x) + \mu_A^3(x) - \mu_A^3(x)\mu_A^3(x)}, \sqrt[3]{\frac{\mu_A^3(x) + \mu_B^3(x)}{2}} \right\}, \right.$$
$$\left. \max\left\{\nu_A(x)\nu_B(x), \sqrt[3]{\frac{\nu_A^3(x) + \nu_B^3(x)}{2}}\right\} | x \in X\right]$$
$$= \left\{ \left\langle x, \sqrt[3]{\frac{\mu_A^3(x) + \mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x) + \nu_B^3(x)}{2}} \right\rangle | x \in X\right\}$$

$$= A@B \tag{4.1}$$

and

$\left((A@B) \to (A \boxplus_F B)^C\right)^C$

$$= \left[x, \min\left\{\sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \sqrt[3]{\mu_A^3(x)+\mu_A^3(x)-\mu_A^3(x)\mu_A^3(x)}\right\}, \max\left\{\sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}, \nu_A(x)\nu_B(x)\right\} |x \in X\right]$$

$$= \left\{\left\langle x, \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}\right\rangle |x \in X\right\}$$

$$= A@B \tag{4.2}$$

From (4.1) and (4.2)⇒ (i) holds.

Thus, $\left((A \boxplus_F B) \to (A@B)^C\right)^C = \left((A@B) \to (A \boxplus_F B)^C\right)^C = (A@B)$.

(iii) Let $\left((A \boxtimes_F B) \to (A@B)^C\right)^C$

$$= \left[x, \min\left\{\mu_A(x)\mu_B(x), \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}\right\}, \max\left\{\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}\right\} |x \in X\right]$$

$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right\rangle |x \in X\right\}$$

$$= A \boxtimes_F B \tag{4.3}$$

and

$\left((A@B) \to (A \boxtimes_F B)^C\right)^C$

$$= \left[x, \min\left\{\sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \mu_A(x)\mu_B(x)\right\}, \max\left\{\sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}, \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right\} |x \in X\right]$$

$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x)+\nu_A^3(x)-\nu_A^3(x)\nu_A^3(x)}\right\rangle |x \in X\right\}$$

$$= A \boxtimes_F B \tag{4.4}$$

From (4.3) and (4.4)⇒ (iii) holds.

Thus, $\left((A \boxtimes_F B) \to (A@B)^C\right)^C = \left((A@B) \to (A \boxtimes_F B)^C\right)^C = (A \boxtimes_F B)$.

(v) Let $\left((A \boxplus_F B) \to (A\#B)^C\right)^C$

$$= \left[x, \min\left\{\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}, \frac{\sqrt[3]{2}\mu_A(x)\mu_B(x)}{\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)}}\right\}, \max\left\{\nu_A(x)\nu_B(x), \frac{\sqrt[3]{2}\nu_A(x)\nu_B(x)}{\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)}}\right\} |x \in X\right]$$

$$= \left\{\left\langle x, \frac{\sqrt[3]{2}\mu_A(x)\mu_B(x)}{\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)}}, \frac{\sqrt[3]{2}\nu_A(x)\nu_B(x)}{\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)}}\right\rangle |x \in X\right\}$$

$$= A\#B \tag{4.5}$$

and

$\left((A\#B) \to (A \boxplus_F B)^C\right)^C$

$$= \left[x, \min\left\{ \frac{\sqrt[3]{2}\mu_A(x)\mu_B(x)}{\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)}}, \sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)} \right\}, \right.$$

$$\left. \max\left\{ \frac{\sqrt[3]{2}\nu_A(x)\nu_B(x)}{\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)}}, \nu_A(x)\nu_B(x) \right\} |x \in X \right]$$

$$= \left\{ \left\langle x, \frac{\sqrt[3]{2}\mu_A(x)\mu_B(x)}{\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)}}, \frac{\sqrt[3]{2}\nu_A(x)\nu_B(x)}{\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)}} \right\rangle |x \in X \right\}$$

$$= A\#B \tag{4.6}$$

From (4.5) and (4.4)⇒ (v)holds.

Thus, $\big((A \boxplus_F B) \to (A\#B)^C\big)^C = \big((A\#B) \to (A \boxplus_F B)^C\big)^C = (A\#B)$.

(vii) Let $\big((A \boxtimes_F B) \to (A\#B)^C\big)^C$

$$= \left[x, \min\left\{ \mu_A(x)\mu_B(x), \frac{\sqrt[3]{2}\mu_A(x)\mu_B(x)}{\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)}} \right\}, \right.$$

$$\left. \max\left\{ \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}, \frac{\sqrt[3]{2}\nu_A(x)\nu_B(x)}{\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)}} \right\} |x \in X \right]$$

$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)} \right\rangle |x \in X \right\}$$

$$= A \boxtimes_F B \tag{4.7}$$

and

$\big((A\#B) \to (A \boxtimes_F B)^C\big)^C$

$$= \left[x, \min\left\{ \frac{\sqrt[3]{2}\mu_A(x)\mu_B(x)}{\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)}}, \mu_A(x)\mu_B(x) \right\}, \right.$$

$$\left. \max\left\{ \frac{\sqrt[3]{2}\nu_A(x)\nu_B(x)}{\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)}}, \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)} \right\} |x \in X \right]$$

$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)} \right\rangle |x \in X \right\}$$

$$= A \boxtimes_F B \tag{4.8}$$

From (4.7) and (4.8)⇒(vii) holds.

Thus, $\big((A \boxtimes_F B) \to (A\#B)^C\big)^C = \big((A\#B) \to (A \boxtimes_F B)^C\big)^C = (A \boxtimes_F B)$.

(ix) Let $\big((A \boxplus_F B) \to (A\$B)^C\big)^C$

$$= \left[x, \min\left\{ \sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}, \sqrt[3]{\mu_A(x)\mu_B(x)} \right\}, \right.$$

$$\left. \max\left\{ \nu_A(x)\nu_B(x), \sqrt[3]{\nu_A(x)\nu_B(x)} \right\} \ |x \in X \right]$$

$$= \left\{ \left\langle x, \sqrt[3]{\mu_A(x)\mu_B(x)}, \sqrt[3]{\nu_A(x)\nu_B(x)} \right\rangle |x \in X \right\}$$

$$= A\$B \tag{4.9}$$

and

$\big((A\$B) \to (A \boxplus_F B)^C\big)^C$

$$= \left[x, \min\left\{ \sqrt[3]{\mu_A(x)\mu_B(x)}, \sqrt[3]{\mu_A^3(x)+\mu_A^3(x)-\mu_A^3(x)\mu_A^3(x)} \right\}, \right.$$

$$\left. \max\left\{ \sqrt[3]{\nu_A(x)\nu_B(x)}, \nu_A(x)\nu_B(x) \right\} \ |x \in X \right]$$

$$= \left\{ \left\langle x, \sqrt[3]{\mu_A(x)\mu_B(x)}, \sqrt[3]{\nu_A(x)\nu_B(x)} \right\rangle |x \in X \right\}$$
$$= A\$B \tag{4.10}$$

From (4.9) and (4.10)⇒(ix) holds.

Thus, $\left((A \boxplus_F B) \to (A\$B)^C\right)^C = \left((A\$B) \to (A \boxplus_F B)^C\right)^C = (A\$B)$.

(xi) Let $\left((A \boxtimes_F B) \to (A\$B)^C\right)^C$

$$= \Bigg[x, \min\left\{ \mu_A(x)\mu_B(x), \sqrt[3]{\mu_A(x)\mu_B(x)} \right\},$$
$$\max\left\{ \sqrt[3]{\nu_A^3(x) + \nu_A^3(x) - \nu_A^3(x)\nu_A^3(x)}, \sqrt[3]{\nu_A(x)\nu_B(x)} \right\} \ |x \in X \Bigg]$$
$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x) + \nu_A^3(x) - \nu_A^3(x)\nu_A^3(x)} \right\rangle |x \in X \right\}$$
$$= A \boxtimes_F B \tag{4.11}$$

and

$\left((A\$B) \to (A \boxtimes_F B)^C\right)^C$

$$= \Bigg[x, \min\left\{ \sqrt[3]{\mu_A(x)\mu_B(x)}, \mu_A(x)\mu_B(x) \right\},$$
$$\max\left\{ \sqrt[3]{\nu_A(x)\nu_B(x)}, \sqrt[3]{\nu_A^3(x) + \nu_B^3(x) - \nu_A^3(x)\nu_B^3(x)} \right\} |x \in X \Bigg]$$
$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x) + \nu_B^3(x) - \nu_A^3(x)\nu_B^3(x)} \right\rangle |x \in X \right\}$$
$$= A \boxtimes_F B \tag{4.12}$$

From (4.11) and (4.12)⇒ (xi) holds.

Thus, $\left((A \boxtimes_F B) \to (A\$B)^C\right)^C = \left((A\$B) \to (A \boxtimes_F B)^C\right)^C = (A \boxtimes_F B)$.

$(xiii)$ Let $\left((A \boxplus_F B) \to (A \boxtimes_F B)^C\right)^C$

$$= \Bigg[x, \min\left\{ \sqrt[3]{\mu_A^3(x) + \mu_B^3(x) - \mu_A^3(x)\mu_B^3(x)}, \mu_A(x)\mu_B(x) \right\},$$
$$\max\left\{ \nu_A(x)\nu_B(x), \sqrt[3]{\nu_A^3(x) + \nu_B^3(x) - \nu_A^3(x)\nu_B^3(x)} \right\} \ |x \in X \Bigg]$$
$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x) + \nu_B^3(x) - \nu_A^3(x)\nu_B^3(x)} \right\rangle |x \in X \right\}$$
$$= A \boxtimes_F B \tag{4.13}$$

and

$\left((A \boxtimes_F B) \to (A \boxplus_F B)^C\right)^C$

$$= \Bigg[x, \min\left\{ \mu_A(x)\mu_B(x), \sqrt[3]{\mu_A^3(x) + \mu_B^3(x) - \mu_A^3(x)\mu_B^3(x)} \right\},$$
$$\max\left\{ \sqrt[3]{\nu_A^3(x) + \nu_B^3(x) - \nu_A^3(x)\nu_B^3(x)}, \nu_A(x)\nu_B(x) \right\} \ |x \in X \Bigg]$$
$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x) + \nu_B^3(x) - \nu_A^3(x)\nu_B^3(x)} \right\rangle |x \in X \right\}$$
$$= A \boxtimes_F B \tag{4.14}$$

From (4.13) and (4.14)⇒(xiii) holds.

Thus, $\left((A \boxplus_F B) \to (A \boxtimes_F B)^C\right)^C = \left((A \boxtimes_F B) \to (A \boxplus_F B)^C\right)^C$
$= (A \boxtimes_F B)$. □

The proof of the following Corollaries follows from Theorem 4.2.

Corollary 4.3. *For* $A, B \in FFS(X)$,
$\big((A \boxtimes_F B) \to (A@B)^C\big)^C = \big((A@B) \to (A \boxtimes_F B)^C\big)^C$
$= \big((A \boxtimes_F B) \to (A\#B)^C\big)^C = \big((A\#B) \to (A \boxtimes_F B)^C\big)^C$
$= \big((A \boxtimes_F B) \to (A\$B)^C\big)^C = \big((A\$B) \to (A \boxtimes_F B)^C\big)^C$
$= \big((A \boxplus_F B) \to (A \boxtimes_F B)^C\big)^C = \big((A \boxtimes_F B) \to (A \boxplus_F B)^C\big)^C$
$= (A \boxtimes_F B)$.

Corollary 4.4. *For* $A, B \in FFS(X)$,
$\big((A \boxplus_F B)^C \to (A@B)\big) = \big((A@B)^C \to (A \boxplus_F B)\big)$
$= \big((A \boxplus_F B)^C \to (A\#B)\big) = \big((A\#B)^C \to (A \boxplus_F B)\big)$
$= \big((A \boxplus_F B)^C \to (A\$B)\big) = \big((A\$B)^C \to (A \boxplus_F B)\big)$
$= \big((A \boxtimes_F B)^C \to (A \boxplus_F B)\big) = \big((A \boxplus_F B)^C \to (A \boxtimes_F B)\big)$
$= (A \boxplus_F B)$.

Theorem 4.5. *For* $A, B \in FFS(X)$,
$\left[\left(A^C \to B\right) \boxplus_F \left(A \to B^C\right)^C\right] @ \left[\left(A^C \to B\right) \boxtimes_F \left(A \to B^C\right)^C\right] = (A@B)$.

Proof. Let $\left[\left(A^C \to B\right) \boxplus_F \left(A \to B^C\right)^C\right]$

$$= \left\{\left\langle x, \sqrt[3]{\mu_A^3(x) + \mu_B^3(x) - \mu_A^3(x)\mu_B^3(x)}, \nu_A(x)\nu_B(x)\right\rangle | x \in X\right\} \tag{4.15}$$

and
$\left[\left(A^C \to B\right) \boxtimes_F \left(A \to B^C\right)^C\right]$

$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x) + \nu_B^3(x) - \nu_A^3(x)\nu_B^3(x)}\right\rangle | x \in X\right\} \tag{4.16}$$

Now with @ of (4.15) and (4.14),
$\left[\left(A^C \to B\right) \boxplus_F \left(A \to B^C\right)^C\right] @ \left[\left(A^C \to B\right) \boxtimes_F \left(A \to B^C\right)^C\right]$

$$= \left[x, \sqrt[3]{\frac{\left(\sqrt[3]{\mu_A^3(x) + \mu_B^3(x) - \mu_A^3(x)\mu_B^3(x)}\right)^3 + \mu_A^3(x)\mu_B^3(x)}{2}}, \right.$$
$$\left. \sqrt[3]{\frac{\nu_A^3(x)\nu_B^3(x) + \left(\sqrt[3]{\nu_A^3(x) + \nu_B^3(x) - \nu_A^3(x)\nu_B^3(x)}\right)^3}{2}} | x \in X\right]$$
$$= \left\{\left\langle x, \sqrt[3]{\frac{\mu_A^3(x) + \mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x) + \nu_B^3(x)}{2}}\right\rangle | x \in X\right\}$$
$$= (A@B).$$
□

Theorem 4.6. *For* $A, B \in FFS(X)$,
$\left[\left(\left(A^C \to B\right) \boxplus_F \left(A \to B^C\right)^C\right) \bigcap \left(\left(A^C \to B\right) \boxtimes_F \left(A \to B^C\right)^C\right)\right]$
$@ \left[\left(\left(A^C \to B\right) \boxplus_F \left(A \to B^C\right)^C\right) \bigcup \left(\left(A^C \to B\right) \boxtimes_F \left(A \to B^C\right)^C\right)\right] = (A@B)$.

Proof. Taking with $\bigcap$ of (4.15) and (4.14), we get
$\left[\left(\left(A^C \to B\right) \boxplus_F \left(A \to B^C\right)^C\right) \bigcap \left(\left(A^C \to B\right) \boxtimes_F \left(A \to B^C\right)^C\right)\right]$

$$= \left[x, \min\left\{\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}, \mu_A(x)\mu_B(x)\right\},\right.$$
$$\left.\max\left\{\nu_A(x)\nu_A(x), \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right\} \ |x \in X\right]$$
$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right\rangle |x \in X\right\} \tag{4.17}$$

Again taking with $\bigcup$ of (4.15) and (4.14),

$$\left[\left(\left(A^C \to B\right) \boxplus_F \left(A \to B^C\right)^C\right) \bigcup \left(\left(A^C \to B\right) \boxtimes_F \left(A \to B^C\right)^C\right)\right]$$
$$= \left[x, \max\left\{\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}, \mu_A(x)\mu_B(x)\right\},\right.$$
$$\left.\min\left\{\nu_A(x)\nu_B(x), \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right\} \ |x \in X\right]$$
$$= \left\{\left\langle x, \sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}, \nu_A(x)\nu_B(x)\right\rangle |x \in X\right\} \tag{4.18}$$

Now wih @ of (4.17) and (4.18),

$$\left[\left(\left(A^C \to B\right) \boxplus_F \left(A \to B^C\right)^C\right) \bigcap \left(\left(A^C \to B\right) \boxtimes_F \left(A \to B^C\right)^C\right)\right]$$
$$@\left[\left(\left(A^C \to B\right) \boxplus_F \left(A \to B^C\right)^C\right) \bigcup \left(\left(A^C \to B\right) \boxtimes_F \left(A \to B^C\right)^C\right)\right]$$
$$= \left[x, \sqrt[3]{\frac{\left(\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}\right)^3 + \mu_A^3(x)\mu_B^3(x)}{2}},\right.$$
$$\left.\sqrt[3]{\frac{\nu_A^3(x)\nu_B^3(x) + \left(\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right)^3}{2}} |x \in X\right]$$
$$= \left\{\left\langle x, \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}\right\rangle |x \in X\right\}$$
$$= A@B.$$

□

Theorem 4.7. *For* $A, B \in FFS(X)$,

$$\left[\left((A \boxplus_F B) \to (A@B)^C\right)^C \bigcup \left((A \boxtimes_F B) \to (A@B)^C\right)^C\right]$$
$$\bigcup \left[\left((A \boxplus_F B) \to (A@B)^C\right)^C \bigcap \left((A \boxtimes_F B) \to (A@B)^C\right)^C\right]$$
$$= A@B.$$

Proof. From Theorem (4.2), we have

$$\left((A \boxplus_F B) \to (A@B)^C\right)^C$$
$$= \left\{\left\langle x, \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}\right\rangle |x \in X\right\} \tag{4.19}$$

and

$$\left((A \boxtimes_F B) \to (A@B)^C\right)^C$$
$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x)+\nu_A^3(x)-\nu_A^3(x)\nu_A^3(x)}\right\rangle |x \in X\right\} \tag{4.20}$$

Now with $\bigcup$ of (4.19) and (4.20),

$$\left[\left((A \boxplus_F B) \to (A@B)^C\right)^C \bigcup \left((A \boxtimes_F B) \to (A@B)^C\right)^C\right]$$

$$= \left[x, \max\left\{\sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, (\mu_A(x)\mu_B(x))\right\},\right.$$
$$\left.\min\left\{\sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}, \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right\} |x \in X\right]$$
$$= \left\{\left\langle x, \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}\right\rangle |x \in X\right\} \tag{4.21}$$

and with $\bigcap$ of (4.19) and (4.20),

$$\left[\left((A \boxplus_F B) \to (A@B)^C\right)^C \bigcap \left((A \boxtimes_F B) \to (A@B)^C\right)^C\right]$$
$$= \left[x, \min\left\{\sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, (\mu_A(x)\mu_B(x))\right\},\right.$$
$$\left.\max\left\{\sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}, \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right\} |x \in X\right]$$
$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right\rangle |x \in X\right\} \tag{4.22}$$

Now we consider,

$$\left[\left((A \boxplus_F B) \to (A@B)^C\right)^C \bigcup \left((A \boxtimes_F B) \to (A@B)^C\right)^C\right]$$
$$\bigcup \left[\left((A \boxplus_F B) \to (A@B)^C\right)^C \bigcap \left((A \boxtimes_F B) \to (A@B)^C\right)^C\right]$$
$$= \left[x, \max\left\{\sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, (\mu_A(x)\mu_B(x))\right\},\right.$$
$$\left.\min\left\{\sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}, \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right\} |x \in X\right]$$
$$= \left\{\left\langle x, \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}\right\rangle |x \in X\right\}$$
$$= A@B.$$

□

Theorem 4.8. *For* $A, B \in FFS(X)$,

$$\left[\left((A \boxplus_F B) \to (A@B)^C\right)^C \bigcup \left((A \boxtimes_F B) \to (A@B)^C\right)^C\right]$$
$$\bigcap \left[\left((A \boxplus_F B) \to (A@B)^C\right)^C \bigcap \left((A \boxtimes_F B) \to (A@B)^C\right)^C\right]$$
$$= A \boxtimes_F B.$$

Proof. The proof is similar to that of Theorem (4.7). □

Theorem 4.9. *For* $A, B \in FFS(X)$,

$$\left((A \boxplus_F B)^C \to (A@B)\right) @ \left((A \boxtimes_F B) \to (A@B)^C\right)^C = (A@B).$$

Proof. Let $\left((A \boxplus_F B)^C \to (A@B)\right)$

$$= \left\{\left\langle x, \sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}, \nu_A(x)\nu_B(x)\right\rangle |x \in X\right\} \tag{4.23}$$

and

$$\left((A \boxtimes_F B) \to (A@B)^C\right)^C$$
$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right\rangle |x \in X\right\} \tag{4.24}$$

Now with @ of (4.23) and (4.24),

$$\left((A \boxplus_F B)^C \to (A@B)\right) @ \left((A \boxtimes_F B) \to (A@B)^C\right)^C$$

$$= \left[x, \sqrt[3]{\frac{\left(\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}\right)^3+(\mu_A(x)\mu_B(x))^3}{2}}, \right.$$
$$\left. \sqrt[3]{\frac{(\nu_A(x)\nu_B(x))^3+\left(\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right)^3}{2}} \,|x\in X\right]$$
$$= \left\{\left\langle x, \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}\right\rangle |x\in X\right\}$$
$$= A@B.$$ □

Theorem 4.10. *For* $A, B \in FFS(X)$,
$\left((A \boxplus_F B)^C \to (A\#B)\right) @ \left((A \boxtimes_F B) \to (A\#B)^C\right)^C = (A@B)$.

Proof. Let $\left((A \boxplus_F B)^C \to (A\#B)\right)$
$$= \left\{\left\langle x, \sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}, \nu_A(x)\nu_B(x)\right\rangle |x\in X\right\} \tag{4.25}$$
and
$\left((A \boxtimes_F B) \to (A\#B)^C\right)^C$
$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right\rangle |x\in X\right\} \tag{4.26}$$
Now with @ of (4.25) and (4.24),
$\left((A \boxplus_F B)^C \to (A\#B)\right) @ \left((A \boxtimes_F B) \to (A\#B)^C\right)^C$
$$= \left[x, \sqrt[3]{\frac{\left(\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}\right)^3+(\mu_A(x)\mu_B(x))^3}{2}}, \right.$$
$$\left. \sqrt[3]{\frac{(\nu_A(x)\nu_B(x))^3+\left(\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right)^3}{2}} \,|x\in X\right]$$
$$= \left\{\left\langle x, \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}\right\rangle |x\in X\right\}$$
$$= A@B.$$ □

Theorem 4.11. *For* $A, B \in FFS(X)$,
$\left((A \boxplus_F B)^C \to (A\$B)\right) @ \left((A \boxtimes_F B) \to (A\$B)^C\right)^C = (A@B)$.

Proof. Let $\left((A \boxplus_F B)^C \to (A\$B)\right)$
$$= \left\{\left\langle x, \sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}, \nu_A(x)\nu_B(x)\right\rangle |x\in X\right\} \tag{4.27}$$
and
$\left((A \boxtimes_F B) \to (A\$B)^C\right)^C$
$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right\rangle |x\in X\right\} \tag{4.28}$$
Now with @ of (4.27) and (4.28),
$\left((A \boxplus_F B)^C \to (A\$B)\right) @ \left((A \boxtimes_F B) \to (A\$B)^C\right)^C$

$$= \left[x, \sqrt[3]{\frac{\left(\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}\right)^3+(\mu_A(x)\mu_B(x))^3}{2}},\right.$$
$$\left.\sqrt[3]{\frac{(\nu_A(x)\nu_B(x))^3+\left(\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right)^3}{2}}\,|x\in X\right]$$
$$= \left\{\left\langle x, \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}\right\rangle |x\in X\right\}$$
$$= A@B.$$
□

Theorem 4.12. *For* $A, B \in FFS(X)$,
$\left((A \boxtimes_F B)^C \to (A \boxplus_F B)\right) @ \left((A \boxplus_F B) \to (A \boxtimes_F B)^C\right)^C = (A@B)$

Proof. Let $\left((A \boxtimes_F B)^C \to (A \boxplus_F B)\right)$

$$= \left\{\left\langle x, \sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}, \nu_A(x)\nu_B(x)\right\rangle |x\in X\right\} \tag{4.29}$$

and

$\left((A \boxplus_F B) \to (A \boxtimes_F B)^C\right)^C$

$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), \sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right\rangle |x\in X\right\} \tag{4.30}$$

Now with @ of (4.29) and (4.30),

$\left((A \boxtimes_F B)^C \to (A \boxplus_F B)\right) @ \left((A \boxplus_F B) \to (A \boxtimes_F B)^C\right)^C$

$$= \left[x, \sqrt[3]{\frac{\left(\sqrt[3]{\mu_A^3(x)+\mu_B^3(x)-\mu_A^3(x)\mu_B^3(x)}\right)^3+(\mu_A(x)\mu_B(x))^3}{2}},\right.$$
$$\left.\sqrt[3]{\frac{(\nu_A(x)\nu_B(x))^3+\left(\sqrt[3]{\nu_A^3(x)+\nu_B^3(x)-\nu_A^3(x)\nu_B^3(x)}\right)^3}{2}}\,|x\in X\right]$$
$$= \left\{\left\langle x, \sqrt[3]{\frac{\mu_A^3(x)+\mu_B^3(x)}{2}}, \sqrt[3]{\frac{\nu_A^3(x)+\nu_B^3(x)}{2}}\right\rangle |x\in X\right\}$$
$$= A@B.$$
□

5. Results and discussion

More importantly, in this paper we have proposed some new operations $[(A@B), (A\$B), (A\#B), (A*B), (A\to B)]$ for FFS and discussed many interesting properties not limit to novel operations $(\boxplus_F, \boxtimes_F, \Box, \Diamond, \cap, \cup)$, which can enrich the operation theory.

6. Conclusion remarks

In this paper, we defined some new operators $[(A@B), (A\$B), (A\#B), (A*B), (A\to B)]$ of Fermatean fuzzy sets. Then we discussed several properties of these operators. Further we proved necessity and possibility operators of Fermatean fuzzy sets. Finally, we have identified and proved several of these properties, particularly those involving the operator $A \to B$ defined as Fermatean fuzzy implication with other operators. Our study prompts for further properties as also for defining possibly new operators.

7. Future scope

Thus there remains scope for studying more properties of these sets arising from those other defining set operations that may be thought of using other ways of combining the functions μ, ν. In further research, we may apply these operators in the field of different areas, for example, dynamic decision and consensus , business and marketing management, design, engineering and manufacturing, information technology and networking applications, human resources management, military applications, energy management, geographic information system applications etc.

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