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SOME INEQUALITIES FOR STATISTICAL (C,1)(H,1)-SUMMABILITY

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ABSTRACT. The concept of statistical summability $C,1)(H,1)$ has recently been introduced by [8]. In this paper we establish some inequalities related to concept of statistical summability $(C,1)(H,1)$.

1. INTRODUCTION

There are several generalizations of usual convergence and the notion of statistical convergence is one of them which was introduced by Fast [5] (see also [6]).

The *natural density* δ of $E \subseteq \mathbb{N}$ is defined by $\delta(E) = \lim\limits_{n} \frac{1}{n}|\{k \leq n : k \in E\}|$, where $\mathbb{N}$ is the set of all natural numbers, and the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\epsilon > 0$, $\delta(\{k : |x_k - L| \geq \epsilon\}) = 0$.

Recently, Acar and Mohiuddine [8] has defined the concept of statistical summability $(C,1)(E,1)$ as follows:

Definition 1.1. Let us write $\tau_n := \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r}x_r$ for a sequence $x = (x_k)$. We say that a sequence $x = (x_k)$ is said to be *statistically summable* $(C,1)(E,1)$ to the number L if the sequence $\tau = (\tau_n)$ is statistically convergent to L, i.e. $st - \lim \tau = L = (CE)_1(st)$-$\lim x$. We denote by $(CE)_1(st)$ the set of all sequences which are statistically summable $(C,1)(E,1)$ and we also call such sequences as statistically $(C,1)(E,1)$-summable sequences. Further, we write $(CE)_1(st)_\infty$ for $(CE)_1(st) \cap l_\infty$.

Let l_∞ and c be the Banach spaces of bounded and convergent sequences of real numbers with the usual supremum norm. Let $A = (a_{nk}), n, k \in \mathbb{N}$, be an infinite matrix of real numbers, and let $x = (x_k)$ be a sequence of real numbers. We write $Ax = (A_n(x))$ if $A_n = \sum\limits_{k} a_{nk}x_k$ converges for each n. Let X and Y be any two sequence spaces. If $x \in X$ implies $Ax \in Y$, then we say that the matrix A maps X into Y. We denote the class of all matrices A which map X into Y by (X,Y). If X and Y are equipped with X-lim and Y-lim, $A \in (X,Y)$ and Y-$\lim Ax = X$-$\lim x$ for all $x \in X$, then we write $A \in (X,Y)_{reg}$.

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In this paper we define $(CE)_1(st)$-conservative matrices and prove some inequalities related to the concepts of $(CE)_1(st)$-conservative matrices which are natural analogues of $(c, st \cap l_\infty)$-matrices (see Kolk [7]). Such type of inequalities are also considered by Çoşkun and [3], Çakan and Altay [1], and Çakan et al [2].

2. Lemmas

We shall need the following lemmas in establishing our results.

Lemma 2.1. *$A \in (c, c)$, that is, A is conservative if and only if*
(i) $||A|| = \sup_n \sum_k |a_{nk}| < \infty$,
(ii) $a_k = \lim_n a_{nk}$, *for each k,*
(iii) $a = \lim_n \sum_k a_{nk}$.
If A is conservative, the number $\chi = \chi(A) = a - \sum_k a_k$ is called the characteristic of A. A is said to be regular if and only if $(i), (ii)$ with $a_k = 0$ for all k; and (iii) with $a = 1$ hold.

The following lemma is an analogue of the above lemma and a consequence of Theorem 1 of Kolk [7].

Lemma 2.2. *$A \in (c, (CE)_1(st)_\infty)$ if and only if*
(i) $\sup_n \sum_k |a_{nk}| < \infty$,
(ii) $(CE)_1(st)\text{-}\lim_n a_{nk} = \alpha_k$ for every k, and
(iii) $(CE)_1(st)\text{-}\lim_n \sum_k a_{nk} = \alpha$.

We call such matrices as $(CE)_1(st)$-*conservative matrices*, and in this case

$$\chi_{(CE)_1(st)} = \alpha - \sum_k \alpha_k$$

is defined which is known as the $(CE)_1(st)$-*characteristic* of A. This number is analogous to the number χ_{st} defined by Çoşkun and Çakan [3].

The following lemma is $(CE)_1(st)$-analogue of a result of Çoşkun and Çakan [3].

Lemma 2.3. *Let $||A|| < \infty$ and $(CE)_1(st)\text{-}\lim_n |a_{nk}| = 0$. Then there exists a $y \in l_\infty$ such that $||y|| \leq 1$ and*

$$(CE)_1(st)\text{-} \limsup \sum_k a_{nk} y_k = (CE)_1(st)\text{-} \limsup \sum_k |a_{nk}|.$$

The following lemma is derived from Lemma 2.3 of Çoşkun and Çakan [3] by replacing st by $(CE)_1(st)$.

Lemma 2.4. *Let A be st-conservative and $\lambda > 0$. Then $(CE)_1(st)\text{-}\limsup_n \sum_k |a_{nk} - \alpha_k| \leq \lambda$ if and only if $(CE)_1(st)\text{-}\limsup_n \sum_k (a_{nk} - \alpha_k)^+ \leq \frac{\lambda+\chi}{2}$ and $(CE)_1(st)\text{-}\limsup_n \sum_k (a_{nk} - \alpha_k)^- \leq \frac{\lambda-\chi}{2}$.*

3. Main results

In this section we establish some inequalities involving the numbers $\chi_{(CE)_1(st)}$, $(CE)_1(st)$-$\limsup x$, $\alpha(x) = (CE)_1(st)$-$\liminf x$, $\limsup x$, and $\liminf x$.

Theorem 3.1. *Let A be conservative and $x \in l_\infty$. Then*

$$\limsup_n \sum_k (a_{nk} - a_k)x_k \leq \frac{\lambda + \chi_{(CE)_1(st)}}{2}\beta(x) + \frac{\lambda - \chi_{(CE)_1(st)}}{2}\alpha(-x) \tag{3.1.1}$$

for some constant $\lambda \geq |\chi_{(CE)_1(st)}|$, if and only if

$$\limsup_n \sum_k |a_{nk} - a_k| \leq \lambda, \tag{3.1.2}$$

$$\lim_n \sum_{k \in E} |a_{nk} - a_k| = 0 \tag{3.1.3}$$

for every $E \subseteq \mathbb{N}$ with $\delta(E) = 0$, where $\beta(x) = (CE)_1(st)$-$\limsup x$ and $\alpha(x) = (CE)_1(st)$-$\liminf x$.

Proof. Necessity. Let $L(x) = \limsup x$ and $l(x) = \liminf x$. Since $\beta(x) \leq L(x)$ and $\alpha(-x) \leq -l(x)$ for all $x \in l_\infty$, we have

$$\limsup_n \sum_k (a_{nk} - a_k)x_k \leq \frac{\lambda + \chi_{(CE)_1(st)}}{2}L(x) - \frac{\lambda - \chi_{(CE)_1(st)}}{2}l(x),$$

and the necessity of (3.1.2) follows from Theorem 1 of Das [4]. Define the matrix $B = (b_{nk})$ by

$$b_{nk} = \begin{cases} a_{nk} - a_k \text{ for } k \in E, \\ 0 \text{ otherwise.} \end{cases}$$

Since A is conservative, using Lemma 2.1, the matrix B satisfies the conditions of Corollary 12 of Simons [9]. Hence there exists a $y \in l_\infty$ such that $||y|| \leq 1$ and

$$\limsup_n \sum_k |b_{nk}| = \limsup_n \sum_k b_{nk}y_k. \tag{3.1.4}$$

Now, let $y = (y_k)$ be defined by

$$y_k = \begin{cases} 1 \text{ for } k \in E, \\ 0 \text{ for } k \notin E. \end{cases}$$

So that, $(CE)_1(st)$-$\lim y = \beta(y) = \alpha(y) = 0$; and by (3.1.1) and (3.1.4) we have

$$\limsup_n \sum_{k \in E} |a_{nk} - a_k| \leq \frac{\lambda + \chi_{(CE)_1(st)}}{2}\beta(y) + \frac{\lambda - \chi_{(CE)_1(st)}}{2}\alpha(-y) = 0$$

and we get (3.1.3).

Sufficiency. Let $x \in l_\infty$. Write $E_1 = \{k : x_k > \beta(x) + \epsilon\}$ and $E_2 = \{k : x_k < \alpha(x) - \epsilon\}$. Then we have $\delta(E_1) = \delta(E_2) = 0$; and hence $\delta(E) = 0$ for $E = E_1 \cap E_2$. We can write

$$\sum_k (a_{nk} - a_k)x_k = \sum_{k \in E}(a_{nk} - a_k)x_k + \sum_{k \notin E}(a_{nk} - a_k)^+ x_k - \sum_{k \notin E}(a_{nk} - a_k)^- x_k,$$

where $\lambda^+ = \max\{0, \lambda\}, \lambda^- = \max\{-\lambda, 0\}$. Hence

$$\begin{aligned}\limsup_n \sum_k (a_{nk} - a_k)x_k &\leq \limsup_n \sum_{k\in E} |a_{nk} - a_k||x_k| + \limsup_n \sum_{k\notin E} (a_{nk} - a_k)^+ x_k \\ &+ \limsup_n [-\sum_{k\notin E} (a_{nk} - a_k)^- x_k] \\ &= I_1(x) + I_2(x) + I_3(x).\end{aligned}$$

From condition (3.1.3), we have $I_1(x) = 0$. Let $\epsilon > 0$, then there is a set E as defined above such that for $k \notin E$,

(3.1.5) $$\alpha(x) - \epsilon < x_k < \beta(x) + \epsilon,\ \beta(-x) - \epsilon < -x_k < \alpha(-x) + \epsilon.$$

Therefore from conditions (3.1.2) and (3.1.5) and Lemma 1 of Das [4], we get

$$I_2(x) \leq \frac{\lambda + \chi_{(CE)_1(st)}}{2}(\beta(x) + \epsilon)$$
$$I_3(x) \leq \frac{\lambda - \chi_{(CE)_1(st)}}{2}(\alpha(-x) + \epsilon).$$

Hence we get

$$\begin{aligned}\limsup_n \sum_k (a_{nk} - a_k)x_k &\leq \frac{\lambda + \chi_{(CE)_1(st)}}{2}\beta(x) + \frac{\lambda - \chi_{(CE)_1(st)}}{2}\alpha(-x) + \lambda\epsilon, \\ &\leq \frac{\lambda + \chi_{(CE)_1(st)}}{2}\beta(x) + \frac{\lambda - \chi_{(CE)_1(st)}}{2}\alpha(-x),\end{aligned}$$

since ϵ was arbitrary. This completes the proof of the theorem.

Theorem 3.2. *Let A be $(CE)_1(st)$-conservative. Then, for some constant $\lambda \geq |\chi_{(CE)_1(st)}|$ and for all $x \in l_\infty$,*

(3.2.1)
$$(CE)_1(st)\text{-}\limsup_n \sum_k (a_{nk} - \alpha_k)x_k \leq \frac{\lambda + \chi_{(CE)_1(st)}}{2}L(x) - \frac{\lambda - \chi_{(CE)_1(st)}}{2}l(x)$$

if and only if

(3.2.2) $$(CE)_1(st)\text{-}\limsup_n \sum_k |a_{nk} - \alpha_k| \leq \lambda.$$

Proof. Necessity. If we define the matrix $B = (b_{nk})$ by $b_{nk} = a_{nk} - \alpha_k$ for all n, k, then, since A is $(CE)_1(st)$-conservative, the matrix B satisfies the hypothesis of Lemma 2.3. Hence we have

$$\begin{aligned}(CE)_1(st)\text{-}\limsup_n \sum_k |b_{nk}| &= (CE)_1(st)\text{-}\limsup_n \sum_k b_{nk}y_k \\ &\leq \frac{\lambda + \chi_{(CE)_1(st)}}{2}L(y) - \frac{\lambda - \chi_{(CE)_1(st)}}{2}l(y) \\ &\leq \left(\frac{\lambda + \chi_{(CE)_1(st)}}{2} + \frac{\lambda - \chi_{(CE)_1(st)}}{2}\right)||y|| \\ &\leq \lambda,\end{aligned}$$

since $||y|| \leq 1$.

Sufficiency. As in Theorem 3.1, for some $k_0 \in \mathbb{N}(k > k_0)$, we can write

$$\sum_k (a_{nk} - a_k)x_k = \sum_{k \le k_0} (a_{nk} - a_k)x_k + \sum_{k > k_0} (a_{nk} - a_k)^+ x_k - \sum_{k > k_0} (a_{nk} - a_k)^- x_k.$$

Since for any $\epsilon > 0$, $l(x) - \epsilon < x_k < L(x) + \epsilon$; and A is $(CE)_1(st)$-conservative, we get by Lemma 2.4 that

$$v(st)\text{-}\limsup_n \sum_k (a_{nk} - \alpha_k)x_k \le (L(x)+\epsilon)\left(\frac{\lambda + \chi_{(CE)_1(st)}}{2}\right) - (l(x)-\epsilon)\left(\frac{\lambda - \chi_{(CE)_1(st)}}{2}\right)$$
$$= \frac{\lambda + \chi_{(CE)_1(st)}}{2} L(x) - \frac{\lambda - \chi_{(CE)_1(st)}}{2} l(x) + \lambda\epsilon,$$

which gives (3.2.1), since ϵ was arbitrary.

Theorem 3.3. *Let A be $(CE)_1(st)$-conservative. Then, for some constant $\lambda \ge |\chi_{(CE)_1(st)}|$ and for all $x \in l_\infty$,*

$$(3.3.1) \quad H(st)\text{-}\limsup_n \sum_k (a_{nk} - \alpha_k) \le \frac{\lambda + \chi_{(CE)_1(st)}}{2}\beta(x) + \frac{\lambda - \chi_{(CE)_1(st)}}{2}\alpha(-x)$$

if and only if (3.2.2) holds and

$$(3.3.2) \qquad (CE)_1(st)\text{-}\lim_n \sum_{k \in E} (a_{nk} - \alpha_k) = 0$$

for every $E \subseteq \mathbb{N}$ with $\delta(E) = 0$.

Proof. Necessity. Let (3.3.1) hold. Since $\beta(x) \le L(x)$ and $-\alpha(x) \le -l(x)$, (3.2.2) follows from Theorem 3.2. Now let us show the necessity of (3.3.2). For any $E \subseteq \mathbb{N}$ with $\delta(E) = 0$, let us define a matrix $B = (b_{nk})$ by the following

$$b_{nk} = \begin{cases} a_{nk} - \alpha_k \text{ for } k \in E, \\ 0 \text{ otherwise.} \end{cases}$$

Then, it is clear that B satisfies the conditions of Lemma 2.3 and hence there exists a $y \in l_\infty$ such that $||y|| \le 1$ and

$$(CE)_1(st)\text{-}\limsup_n \sum_k b_{nk}y_k = (CE)_1(st)\text{-}\limsup_n \sum_k |b_{nk}|.$$

Let us define the sequence $y = (y_k)$ by

$$y_k = \begin{cases} 1 \text{ for } k \in E, \\ 0 \text{ for } k \notin E. \end{cases}$$

Using the fact that $(CE)_1(st)\text{-}\lim y = \beta(y) = \alpha(y) = 0$ and (2.3.1), we get

$$(CE)_1(st)\text{-}\limsup_n \sum_{k \in E} |a_{nk} - \alpha_k| x_k \le \frac{\lambda + \chi_{(CE)_1(st)}}{2}\beta(y) + \frac{\lambda - \chi_{(CE)_1(st)}}{2}\alpha(-y) = 0,$$

and hence we get (3.3.2).

Sufficiency. Let (3.2.2) and (3.3.2) hold and $x \in l_\infty$. As in Theorem 3.1, we can write

$$\sum_k (a_{nk} - \alpha_k)x_k = \sum_{k \in E} (a_{nk} - \alpha_k)x_k + \sum_{k \notin E} (a_{nk} - \alpha_k)^+ x_k - \sum_{k \notin E} (a_{nk} - \alpha_k)^- x_k.$$

Using Lemma 2.4 and Lemma 2.2 for $(CE)_1(st)$-conservativeness of A, we have

$$(CE)_1(st)\text{-}\limsup_n \sum_k (a_{nk}-\alpha_k) \leq \frac{\lambda+\chi_{(CE)_1(st)}}{2}\beta(x)+\frac{\lambda-\chi_{(CE)_1(st)}}{2}\alpha(-x)+\lambda\epsilon.$$

But ϵ was arbitrary, so (3.3.1) holds.

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